**Exercise for Talk 6: Arithmetic compactification of Hilbert modular varieties.**

More carefully discuss the degeneration of elliptic curves at the cusp, following Mumford's original approach. Then we move to the Hilbert modular varieties. Mostly follow Ching-Li Chai's appendix to Wiles' Iwasawa Main Conjecture paper.

**Problem 6.1** (A simple case of log Gauss–Manin connection)**.** In fact, the Gauss–Manin construction works equally well for "log-schemes". We will just discuss a baby case of this. Let $k$ be a field of characteristic zero. Consider a smooth surface $Y$ relative to a smooth curve $X$ over $k$ in the following picture:

$$\begin{array}{ccccc} V & \subset & Y & \hookleftarrow & D = D_1 \cup D_2 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ U & \subset & X & \hookleftarrow & \{0\} \end{array}$$

where both squares are Cartesian, $U = X\backslash\{0\}$ (and thus $V = Y\backslash D$), $f|_V$ is a proper and smooth relative curve, and $D = f^{-1}(\{0\})$ is the union of two smooth (proper) divisor $D_1$ and $D_2$ of $Y$. Assume that in the formal neighborhood at each intersection points of $D_1$ and $D_2$, $f$ is given by $\operatorname{Spec} k[\![x, y, z]\!]/(yz - x) \to \operatorname{Spec} k[\![x]\!]$, and $x = 0$ corresponds to $\{0\} \in X$ and $D_1$ and $D_2$ are given by $V(x, y)$ and $V(x, z)$ respectively.

(In fancy language, this is a typical situation where the log scheme $(Y, D) \to (X, \{0\})$ is log-smooth.)

(1) Consider the sheaf of differential forms with log poles

$$\Omega^1_X(\log\{0\}), \quad \Omega^1_Y(\log D), \quad \Omega^2_Y(\log D) := \wedge^2\Omega^1_Y(\log D)$$

as were defined in Problem 3.2. Show that the natural map

$$f^*\Omega^1_X(\log\{0\}) \to \Omega^1_Y(\log D) \tag{6.1.1}$$

is injective and the cokernel $\Omega^1_{Y/X}(\log D)$ is locally free of rank 1.

(2) Consider the relative de Rham cohomology

$$\mathcal{H}^\bullet_{\mathrm{dR,log}}(Y/X) := R^\bullet f_*\big(\mathcal{O}_Y \to \Omega^1_{Y/X}(\log D)\big).$$

Imitate the solution of Problem 3.3 to prove that there is a natural log-connection

$$\nabla_{\mathrm{GM}} : \mathcal{H}^\bullet_{\mathrm{dR,log}}(Y/X) \longrightarrow \mathcal{H}^\bullet_{\mathrm{dR,log}}(Y/X) \otimes \Omega^1_{X/k}(\log\{0\}).$$

(3) It seems to be reasonable to expect that $\mathcal{H}^\bullet_{\mathrm{dR,log}}(Y/X)$ is locally free over $X$, but we do not know of a reference. (Note that this does not follow from the existence of Gauss–Manin connection, as it has a log pole.) For our later application, we content ourselves to prove that if the dimensions of $H^\bullet(D, \mathcal{O}_D \to \Omega^1_D(\log(D_1 \cap D_2)))$ are the same as $H^\bullet_{\mathrm{dR}}(Y_x)$ for any fiber $x \in X\backslash\{0\}$ at all degrees, then $\mathcal{H}^\bullet_{\mathrm{dR,log}}(Y/X)$ is locally free over $X$.

(4) We know that at each $x \in X\backslash\{0\}$, the natural cup product pairing $H^1_{\mathrm{dR}}(Y_x) \times H^1_{\mathrm{dR}}(Y_x) \to H^2_{\mathrm{dR}}(Y_x)$ is perfect. Suppose that the conditions in (3) holds and the cup product condition holds for $H^\bullet(D, \mathcal{O}_D \to \Omega^1_D(\log(D_1 \cap D_2)))$ as well (for which it seems to be automatic), then the natural cup product

$$\mathcal{H}^1_{\mathrm{dR,log}}(Y/X) \times \mathcal{H}^1_{\mathrm{dR,log}}(Y/X) \longrightarrow \mathcal{H}^2_{\mathrm{dR,log}}(Y/X)$$

is perfect.

**Problem 6.2** (Extension of the de Rham cohomology to the boundary of modular curve). Let $Y$ denote the open modular curve, $X$ its compactification, and $C$ the cusp, all over $\mathbb{Q}$. We have discussed $0 \to \omega \to H^1_{\mathrm{dR}}(\mathcal{E}/Y) \to \omega^{-1} \to 0$ over $Y$. Now, we discuss how one extends the sheaf of differentials and relative de Rham cohomology to the boundary.

Recall Mumford's construction of degeneration of elliptic curves "$\mathbb{G}_m/q^{\mathbb{Z}}$" from $\mathbb{Q}((q))$ to $\mathbb{Q}[\![q]\!]$. The first step is consider a big ring $\mathscr{R} := \mathbb{Q}((q))[\mathfrak{X}^{\pm 1}][\theta]$ (so that $\operatorname{Spec}\mathscr{R}$ is a line bundle over $\mathbb{G}_{m,\mathbb{Q}((q))}$). We take the two associated $\mathbb{Z}$-modules $X$ and $Y$ to be simply $X = Y = \mathbb{Z}$, and the polarization $\phi : Y \to X$ to be the identity map.

Consider the graded algebra

$$R_\phi = \mathbb{Q}[\![q]\!]\left[\begin{array}{c} \theta,\, S(\theta) = q\mathfrak{X}^2\theta,\, \dots,\, S^k(\theta) = q^{k^2}\mathfrak{X}^{2k}\theta \\ \mathfrak{X}\theta,\, S(\mathfrak{X}\theta) = q^2\mathfrak{X}^3\theta,\, S^{-1}(\mathfrak{X}\theta) = \mathfrak{X}^{-1}\theta,\, \dots,\, S^k(\mathfrak{X}\theta) = q^{k^2+k}\mathfrak{X}^{2k+1}\theta \end{array}\right],$$

with an automorphism:

$$S(q) = q, \quad S(\mathfrak{X}) = q\mathfrak{X}, \quad \text{and} \quad S(\theta) = q\mathfrak{X}^2\theta.$$

Then the projective scheme associated to $R_\phi$ admits charts (check!):

$$\begin{aligned} U_0 &= \operatorname{Spec} R_\phi[\tfrac{1}{\theta}]^{\deg=0} = \operatorname{Spec}\mathbb{Q}[\![q]\!][\mathfrak{X}, \mathfrak{X}^{-1}] &&\longleftrightarrow \quad \text{“}|\mathfrak{X}| = 1\text{”} \\ U_1 &= \operatorname{Spec} R_\phi[\tfrac{1}{\mathfrak{X}\theta}]^{\deg=0} = \operatorname{Spec}\mathbb{Q}[\![q]\!][q\mathfrak{X}, \mathfrak{X}^{-1}] &&\longleftrightarrow \quad \text{“}|\mathfrak{X}| \in [1, |q|^{-1}]\text{”} \\ U_2 &= \operatorname{Spec} R_\phi[\tfrac{1}{q\mathfrak{X}^2\theta}]^{\deg=0} = \operatorname{Spec}\mathbb{Q}[\![q]\!][q\mathfrak{X}, (q\mathfrak{X})^{-1}] &&\longleftrightarrow \quad \text{“}|\mathfrak{X}| = |q|^{-1}\text{”} \end{aligned}$$

(1) Write out the general formula for $U_k = \operatorname{Spec} R_\phi\big[\frac{1}{q^{\lfloor k^2/4\rfloor}\mathfrak{X}^k\theta}\big]^{\deg=0}$, and check that $S$ sends $U_k$ isomorphically to $U_{k+2}$. Check also that $U_{2k}$ is an open subscheme of $U_{2k-1}$ and $U_{2k+1}$.

(2) Consider the formal scheme given by completion of $P := \operatorname{Proj}(R_\phi)$ along its special fiber, denoted by $\mathfrak{P}$. Denote the completion of each affine chart by $\mathfrak{U}_k$; they are affine formal schemes associated to Tate algebras of the form

$$\mathbb{Q}[\![q]\!]\langle q^n\mathfrak{X}, (q^{n-1}\mathfrak{X})^{-1}\rangle := \left\{\begin{array}{c} \sum_{m_1,m_2\geq 0} a_{m_1,m_2}(q^n\mathfrak{X})^{m_1}(q^{1-n}\mathfrak{X}^{-1})^{m_2} \text{ with} \\ a_{m_1,m_2} \in \mathbb{Q}[\![q]\!] \text{ and } v_q(a_{m_1,m_2}) \to 0 \text{ as } m_1, m_2 \to \infty \end{array}\right\}$$

Taking the quotient of $\mathfrak{P}$ by the $S$-action is equivalent to glue the (formal) subscheme $\mathfrak{U}_0$ and $\mathfrak{U}_2$ of $\mathfrak{U}_1$ along the isomorphism $S$. It might be less confusing to modify this gluing slightly (after base change to $\mathbb{Q}[\![q^{1/2}]\!]$, and take a different integral model), and write

$$\begin{aligned} \mathfrak{U}_{1,\mathbb{Q}[\![q^{1/2}]\!]} \leftarrow \mathfrak{U}_{1,-} \textstyle\bigcup_{\mathfrak{U}_{1,\circ}} \mathfrak{U}_{1,+}, \quad &\text{where} \quad \mathfrak{U}_{1,-} = \operatorname{Spf}\mathbb{Q}[\![q^{1/2}]\!][\mathfrak{X}^{-1}, q^{1/2}\mathfrak{X}], \\ \mathfrak{U}_{1,\circ} = \operatorname{Spf}\mathbb{Q}[\![q^{1/2}]\!][q^{1/2}\mathfrak{X}, (q^{1/2}\mathfrak{X})^{-1}], \quad &\mathfrak{U}_{1,+} = \operatorname{Spf}\mathbb{Q}[\![q^{1/2}]\!][q\mathfrak{X}, (q^{1/2}\mathfrak{X})^{-1}] \end{aligned}$$

Now, the situation is to glue $\mathfrak{U}_{1,-}$ and $\mathfrak{U}_{1,+}$ along their open subschemes $S : \mathfrak{U}_0 \xrightarrow{\cong} \mathfrak{U}_2$ and $\mathfrak{U}_{1,\circ}$.

Let $\mathfrak{Q}$ denote the result of the gluing, which can be algebraized to a projective variety $Q$ proper over $S := \operatorname{Spec}\mathbb{Q}[\![q^{1/2}]\!]$.

(3) Show that we are in the situation similar to the previous problem; in particular, the fiber over $q = 0$ is isomorphic to the union of special fibers of $\mathfrak{U}_{1,\circ}$ and $\mathfrak{U}_0 \cong \mathfrak{U}_2$, denoted by $D_\circ$ and $D_{02}$. Let $f : Q \to S := \operatorname{Spec}\mathbb{Q}[\![q^{1/2}]\!]$ be the natural projection.

Show that the natural map

$$f^*\Omega^1_S(\log(q = 0)) \to \Omega^1_Q(\log(D_\circ \cup D_{02}))$$

is injective with quotient $\Omega^1_Q(\log)$, locally free of rank 1.

(4) Show that the conditions in Problem 6.1(3) and (4) holds for this map, and thus we deduce an extension of the relative differential sheaf $\omega := f_*\Omega^1_{Q/S}(\log)$ and the relative de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(Q/S)$ to the boundary, satisfying

$$0 \to \omega \to \mathcal{H}^1_{\mathrm{dR}}(Q/S) \to \omega^{-1} \to 0.$$

(5) Using the Gauss–Manin connection to deduce a natural homomorphism

$$\omega^{\otimes 2} \to \Omega^1_S(\log(q=0)).$$

**Problem 6.3** (Light introduction to toric geometry)**.** A classic textbook on this is by Fulton, Introduction to toric varieties.

Let $k$ be a field. A toric variety over $k$ is an algebraic variety $X$ over $k$ containing $\mathbb{G}^n_{m,k}$ for some $n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety. Typical examples includes $\mathbb{P}^n$, $\mathbb{A}^n$, and etc.

A good way to describe toric varieties is the following. Consider the lattice $\mathbb{Z}^n$ inside $\mathbb{R}^n$. The group algebra $k[\mathbb{Z}^n] \cong k[X_1^{\pm 1}, \dots, X_n^{\pm n}]$ corresponds to the torus $\mathbb{G}^n_m$. However, if we take $\mathbb{Z}^n_{\geq 0}$, a submonoid, its associated algebra $k[\mathbb{Z}^n_{\geq 0}] \cong k[X_1, \dots, X_n]$ corresponds to the affine space $\mathbb{A}^n$. This way, we get a functor

$$\begin{array}{ccc} \big\{\text{Submonoids of } \mathbb{Z}^n\big\}^{\mathrm{op}} & \longrightarrow & \mathbf{Sch}_k \\ \sigma & \longrightarrow & \operatorname{Spec} k[\sigma]. \end{array}$$

But a caveat is that this is a contravariant functor. It turns out that starting from the dual side is better. For a strongly convex rational polyhedral cone $\check\sigma \subset (\mathbb{Z}^n)^* \otimes \mathbb{R}$ which is a $\mathbb{R}_{>0}$-span of vectors with rational coefficients and does not contain a straight line through the origin, we define its dual

$$\sigma = \big\{x \in \mathbb{Z}^n;\ \langle x, y\rangle_{\mathbb{R}} > 0 \text{ for every } y \in \check\sigma\big\}$$

Now we have a functor

$$\begin{array}{ccccc} \left\{\begin{matrix}\text{Strongly convex rational polyhedral} \\ \text{cone in } (\mathbb{Z}^n)^*\otimes\mathbb{R}\end{matrix}\right\} & \longrightarrow & \big\{\text{Submonoids of } \mathbb{Z}^n\big\}^{\mathrm{op}} & \longrightarrow & \mathbf{Sch}_k \\ \check\sigma & \longmapsto & \sigma & \longrightarrow & \operatorname{Spec} k[\sigma] =: U_{\check\sigma}. \end{array}$$

(1) Show that if $\check\sigma = \mathbb{R}_{>0}e_1 + \mathbb{R}_{>0}e_2 \subset (\mathbb{Z}^3)^* \otimes \mathbb{R}$, then $\sigma = \{(x,y,z) \in \mathbb{Z}^3;\ x \geq 0, y \geq 0\}$ and $\operatorname{Spec} k[\sigma] = k[X, Y, Z^{\pm}]$. Generalize this to $\check\sigma = \mathbb{R}_{>0}e_1 + \dots + \mathbb{R}_{>0}e_r \subset (\mathbb{Z}^n)^* \otimes \mathbb{R}$.

More generally If $\check\sigma$ is generated by linearly independent vertices $v_1, \dots, v_r$ such that $\check\sigma \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}v_1 \oplus \dots \oplus \mathbb{Z}_{\geq 0}v_r$, show that $\operatorname{Spec} k[\sigma] \cong \mathbb{G}^r_m \times \mathbb{G}^{n-r}_m$.

(2) Moving on to gluing. If $\check\tau$ is a facet of $\check\sigma$, say the intersection with a (rational) hyperplane, defined by $\underline{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Moreover, assume that $\langle \underline{a}, \check\sigma\rangle \geq 0$. Show that $U_{\check\tau}$ is an open subspace of $U_{\check\sigma}$, and more precisely, $k[\check\tau] = k[\check\sigma][(X_1^{a_1} \cdots X_n^{a_n})^{-1}]$.

Now, consider a simple case: $(\mathbb{Z})^* \otimes \mathbb{R}$ can be decomposed into $\mathbb{R}_{\geq 0}, \{0\}, \mathbb{R}_{\leq 0}$. Each corresponds to $\operatorname{Spec} k[X], \operatorname{Spec} k[X^{\pm}], \operatorname{Spec} k[X^{-1}]$. When gluing them together, we get $\mathbb{P}^1_k$.

Basically, if we have a rational polyhedral cone in $(\mathbb{Z}^n)^* \otimes \mathbb{R}$, decomposed into strongly convex ones, then we can consider the schemes associated to the strongly convex polyhedral cones and their facets, and glue them together by gluing along the facets = open subsets.

Try some examples of such glueings. E.g. decompose $(\mathbb{Z}^2)^* \otimes \mathbb{R}$ into $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $\{(a,b);\ a \leq 0, a+b \geq 0\}$, and $\{(a,b);\ b \leq 0, a+b \leq 0\}$. Show that this cone decomposition gives rise to $\mathbb{P}^2$.

In general, if we decompose the whole $(\mathbb{Z}^n)^* \otimes \mathbb{R}$ into strongly convex rational polyhedral cones, you will get a proper toric variety.

**Problem 6.4** (Koecher principle for Hilbert modular forms)**.** Here is the argument for Koecher principle for Hilbert modular forms. A reference is Mladen Dimitrov, Compactifications arithmétiques des variétés de Hilbert et formes modulaires de Hilbert pour $\Gamma_1(\mathfrak{c}, n)$.

The statement goes as follows: let $(\mathfrak{c}, \mathfrak{c}^+)$ denote an ideal of a totally real field $F$ (with $[F : \mathbb{Q}] \geq 2$), and let $N \geq 3$ be an integer. Let $\mathcal{M}_\mathfrak{c}(N)$ denote the moduli space of Hilbert modular varieties (with $\mathcal{O}_F$-endomorphism and $(\mathfrak{c}, \mathfrak{c}^+)$-polarizations). Let $\kappa = ((k_\tau)_\tau, w)$ denote a paritious weight. Fixing a datum that defines a toroidal compactification of $\mathcal{M}_\mathfrak{c}(N)$, namely $\overline{\mathcal{M}}_\mathfrak{c}(N)$. (Let $F'$ denote the field obtained by adjoining the Galois closure of $F$ by some square roots of some units in $\mathcal{O}_F$. This is the field where each individual automorphic line bundle is defined.) Then we have for every $\mathcal{O}_{F'}[\frac{1}{N}]$-algebra $R$, we have

$$H^0(\mathcal{M}_\mathfrak{c}(N), \omega^\kappa) = H^0(\overline{\mathcal{M}}_\mathfrak{c}(N), \omega^\kappa).$$

The argument goes as follows. Take a modular form form the left hand side, write the $q$-expansion near a cusp labeled by $(\mathfrak{a}, \mathfrak{b}, H, \beta, i_N)$, then we have

$$f(q) = \sum_{\xi \in \mathfrak{ab}} a_\xi q^\xi.$$

One of the key properties of the $q$-expansion is that $a_{u^2\xi} = a_\xi$ for every $u \in \mathcal{O}_F^\times$. But we want to show that $a_\xi = 0$ unless $\xi$ is totally positive. Suppose this is not true, say $a_{\xi_0} \neq 0$ for $\xi_0$ not totally positive. Then there exists $\xi_0^* \in (M^*)^+$ such that $\mathrm{Tr}_{F/\mathbb{Q}}(\xi_0^* \xi_0) < 0$ (as this is how we naturally identify $M^*$ with $M^\vee$).

(1) Prove that there exists units $u \in \mathcal{O}_F^\times$ such that $\mathrm{Tr}_{F/\mathbb{Q}}(u^2 \xi_0^* \xi_0) \to -\infty$.

This is a problem. If $\check{\sigma}$ is one of the cones when decomposing $(M^*)^+$ into $\mathcal{O}_F$-stable cones. Then by definition, $f(q)$ needs to be a meromorphic function over $\operatorname{Spec} R_\sigma$, where

$$R_\sigma = R[\![q^\zeta;\ \zeta \in M \text{ such that } \langle \zeta, x \rangle \text{ for all } x \in \check{\sigma}]\!].$$

(2) From this deduce a contradiction.

**Hints.**

Problem 6.1 (1) It is enough to work at the completion at each closed point of $Y$. The most non-trivial case happens at the intersection of $D_1$ and $D_2$. Using the local model given in the condition, we can write down the map (6.1.1) as

$$\frac{k[\![x,y,z]\!]}{(yz-x)}\cdot\frac{dx}{x}\longrightarrow\frac{k[\![x,y,z]\!]}{(yz-x)}\cdot\frac{dy}{y}\oplus\frac{k[\![x,y,z]\!]}{(yz-x)}\cdot\frac{dz}{z}$$

But from the relation $yz = x$, we deduce that $\frac{dx}{x} = \frac{dy}{y} + \frac{dz}{z}$. The statement is clear.

Problem 6.2: The description of Mumford's construction can be found in Mumford's paper An analytic construction of degenerating abelian varieties over complete rings. section 5. In this paper, Mumford wrote at the introduction: "Instead I conclude the paper with many examples. For me, one of the most enjoyable features of this research was the beauty of the examples which one works out without a great deal of extra effort." So we suggest to read the example in section 6 of that paper.

Problem 6.4: (2) For a function of the form $f(q) = \sum a_\xi q^\xi$ to be a rational function on $R_{\check{\sigma}}$, it is necessary that for every $\xi^* \in \check{\sigma}$, number of $\xi$'s such that $a_\xi \neq$ and $\mathrm{Tr}_{F/\mathbb{Q}}(\xi\xi^*) < 0$ is finite.

Yet the $\mathcal{O}_F$-invariant condition says that $a_{u^2\xi_0} = a_{\xi_0} \neq 0$. but our choice of $u$ and $\xi_0$ ensures that $\mathrm{Tr}_{F/\mathbb{Q}}(u^2\xi_0\xi_0^*) \to -\infty$. This is the contraction we need.