Exercise for Talk 7: General theory of Shimura varieties.

We will introduce basic theory for Shimura varieties, the theory of Shimura reciprocity law. Classical references include Milne's notes, Deligne's Bourbaki talk, but they are not so easy to understand.

Problem 7.1 (h versus μ)**.** Let T be a torus over $\mathbb{R}$. Show that there is a one-to-one correspondence between

$$\big\{\text{homomorphisms } h : \mathbb{S} \to T\big\} \longleftrightarrow \big\{\text{homomorphisms } \mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}\big\}$$
$$h \longmapsto \mu_h$$

where $\mu_h : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z\mapsto(z,1)} \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} T_{\mathbb{C}}$.

Problem 7.2 (Pure inner forms)**.** We introduce the concept of inner forms and pure inner forms. Let G be an algebraic group over $\mathbb{Q}$, a *form* G' of G is an algebraic group over $\mathbb{Q}$ such that $G' \times_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq G \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. (From this definition, it is clear that a form of a form is a form.)

(1) Show that the set of isomorphism classes of forms of G is the same as the cohomology group $H^1(\mathbb{Q}, \mathrm{Aut}(G_{\overline{\mathbb{Q}}}))$, where $\mathrm{Aut}(G_{\overline{\mathbb{Q}}})$ is the automorphism group of $G_{\overline{\mathbb{Q}}}$ which carries a natural action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

(2) There is a natural class of automorphisms of $G_{\overline{\mathbb{Q}}}$ induced by conjugation, or more precisely, conjugation by elements in $G_{\mathrm{ad}}(\overline{\mathbb{Q}})$. An *inner form* of G is a cohomology class $H^1(\mathbb{Q}, G_{\mathrm{ad}}(\overline{\mathbb{Q}}))$.

Sometimes, we prefer to see the conjugation by an element truly from $G(\overline{\mathbb{Q}})$. An *pure inner form* of G is a cohomology class in $H^1(\mathbb{Q}, G(\overline{\mathbb{Q}}))$.

Given a class $(\sigma \mapsto c_\sigma)$ in either $H^1(\mathbb{Q}, G_{\mathrm{ad}}(\overline{\mathbb{Q}}))$ or $H^1(\mathbb{Q}, G(\overline{\mathbb{Q}}))$, we can construct a concrete form G' as follows. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ act on $G_{\overline{\mathbb{Q}}}$ via $g \mapsto \mathrm{Ad}_{c_\sigma}(\sigma(g))$. Show that this is indeed an action and hence defines by Galois descent an algebraic group over $\mathbb{Q}$.

(3) Can you give examples for which the natural maps

$$H^1(\mathbb{Q}, G(\overline{\mathbb{Q}})) \to H^1(\mathbb{Q}, G_{\mathrm{ad}}(\overline{\mathbb{Q}})) \to H^1(\mathbb{Q}, \mathrm{Aut}(G_{\overline{\mathbb{Q}}}))$$

are not isomorphisms. We often use languages such as: image of $H^1(\mathbb{Q}, G(\overline{\mathbb{Q}}))$ is precisely the forms that can be lifted to pure inner forms.

(4) Show that a pure inner form of a pure inner form is a pure inner form (the same is also true for inner forms).

(5) Back to the story about Shimura varieties, recall that (SV2) implies that conjugation by $h(i)$ is a Cartan involution on G_{ad}. Then $G_{\mathrm{ad},c} := \{g \in G_{\mathrm{ad}}(\mathbb{C});\ h(i)gh(i)^{-1} = \bar{g}\}$ is compact. Show that $G_{\mathrm{ad},c}$ is an inner form of G.

(6) Assume that G_c is semisimple real group whose $\mathbb{R}$-points are compact. Show that there is a one-to-one correspondence between

$$\coprod_{G' \in H^1(\mathbb{R}, G_c)} \big\{G'(\mathbb{R})\text{-conj classes of homomorphism } \mathbb{S} \to G'_{\mathbb{R}} \text{ satisfying (SV1)(SV2)}\big\}$$
$$\cong \big\{G_c(\mathbb{C})\text{-conjugacy classes of minuscule cocharacters } \mu : \mathbb{G}_{m,\mathbb{C}} \to G_{c,\mathbb{C}}\big\}$$
$$\cong \big\{G_c(\mathbb{R})\text{-conjugacy classes of characters } h : \mathbb{S} \to G_c \text{ satisfying (SV1)}\big\}.$$

Problem 7.3 (Shimura set associated to CM types)**.** Let E be a CM field with F its maximal totally real subfield. Recall that a CM type is a set of embeddings $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ such that

$\mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C}) = \Phi \sqcup \Phi^c$, where $\Phi^c := \{c \circ \phi;\ \phi \in \Phi\}$ and c denotes the complex conjugation. Consider the torus $T := \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$. It comes equipped with a cocharacter

$$\begin{aligned} \mu_\Phi : \mathbb{G}_{m,\mathbb{C}} &\longrightarrow T_{\mathbb{C}} \cong \textstyle\prod_{\phi\in\Phi} \mathbb{G}_{m,E} \times_{E,\phi} \mathbb{C} \\ z &\longmapsto z \text{ at each } \phi \in \Phi. \end{aligned}$$

The group T admits a subgroup $T^{\mathbb{Q}}$ whose R-points for a $\mathbb{Q}$-algebra R is

$$T^{\mathbb{Q}}(R) = \big\{x \in T(R) = (R \otimes_{\mathbb{Q}} E)^\times;\ \mathrm{Nm}_{E/F}(x) \in R^\times\big\}.$$

(1) Observe that μ_Φ has image in $T^{\mathbb{Q}}$.

(2) By the previous problem, μ_Φ corresponds to $h_\Phi : \mathbb{S} \to T_{\mathbb{R}}$ (or even $h_\Phi^{\mathbb{Q}} : \mathbb{S} \to T_{\mathbb{R}}^{\mathbb{Q}}$)

(3) Show that the reflex field E_Φ of $(T,\{h_\Phi\})$ or $(T^{\mathbb{Q}},\{h_\Phi^{\mathbb{Q}}\})$ can be described as follows: let $\mathbb{Q}^{\mathrm{alg}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let H denote the subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ that stabilizers the CM type Φ, that is for any $h \in H$, $\{h \circ \phi;\ \phi \in \Phi\} = \Phi$. Then E_Φ is the subfield of $\mathbb{Q}^{\mathrm{alg}}$ fixed by H.

(4) Take a special case: $E = E_0F$ for E_0 an imaginary quadratic field and F a totally real field. Fix one embedding $\tau : E_0 \to \mathbb{C}$. Show that this induces a CM type $\Phi_\tau := \{\phi \in \mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C});\ \phi|_{E_0} = \tau\}$. Show that the reflex field of this Φ_τ is just E_0. What's the corresponding Shimura reciprocity map?

Problem 7.4 (Computation of the reflex field of a special type of Shimura curve)**.** This type of Shimura curve appears in the study of generalizations of Heegner points to the totally real case.

Let F be a totally real field, and let B be a quaternion algebra over F such that there is a unique $\tau_0 : F \to \mathbb{R}$:

$$B \otimes_{F,\tau} \mathbb{R} \cong \begin{cases} \mathrm{M}_2(\mathbb{R}) & \tau = \tau_0 \\ \mathbb{H} & \tau \neq \tau_0 \end{cases}$$

Let $G = \mathrm{Res}_{F/\mathbb{Q}}B^\times$. Then we can define a Shimura datum for G, by taking h to be the $G(\mathbb{R})$-conjugacy class of

$$\begin{aligned} h : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times &\longrightarrow G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \textstyle\prod_{\tau\neq\tau_0} \mathbb{H}^\times \\ z = x + iy &\longmapsto \Big(\big(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\big), 1, \dots, 1\Big). \end{aligned}$$

Show that the reflex field of this Shimura datum is F embedded in $\mathbb{C}$ via τ_0, precisely the one that we used above.

(The upshot is that the Shimura curve is then defined over F embedded in $\mathbb{C}$ via τ_0. Somehow, one should intrinsically think of this Shimura curve defined over F canonically, and associated to B intrinsically. Namely, if we change how F embeds into $\mathbb{C}$, it will affect accordingly how the Shimura curve over F is embedded in $\mathbb{C}$.)

Problem 7.5 (CM theory)**.** The Shimura reciprocity map comes from Complex Multiplication Theory. We only aim to explain the relation in details, but leave the key computation to later exercises. To state it, we consider the following. Let E be a CM field and Φ a CM type. Assume that A is an abelian variety with endomorphism $\mathcal{O}_E$ and CM type Φ (meaning, if we look at the tangent space $T_0(A)$ of A at the origin, which carries an action of structure sheaf $\mathbb{C}$ and an $\mathcal{O}_E$-action, then

$$T_0(A) \cong \prod_{\phi\in\Phi} \mathbb{C}$$

such that $\mathcal{O}_E$ acts on the ϕ-factor via the embedding $\phi \in \Phi$.)

We all know that for all such A, the complex points are isomorphic to $\mathbb{C}^g/\mathfrak{a}$ for an $\mathcal{O}_E$-ideal $\mathfrak{a}$, or more precisely, the quotient $\big(\prod_{\phi\in\Phi} E \otimes_{E,\phi} \mathbb{C}\big)/\mathfrak{a}$, to indicate the $\mathcal{O}_E$-action on each factor $\mathbb{C}$. Since there are only finitely many such abelian variety up to isomorphism (corresponding to the ideal class group of E), they must be defined over a number subfield of $\mathbb{C}$.

Let $\mathbb{Q}^{\mathrm{alg}}$ denote the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. The goal is to understand $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/E_\Phi)$ on the set of CM abelian varieties of CM type Φ, and their division points (and see that Shimura reciprocity map shows up this way.)

(1) Show that for every $\tau \in \mathrm{Gal}(\mathbb{C}/E_\Phi)$, by looking at the action of E on the tangent space at 0, show that $\tau A := A \otimes_{\mathbb{C},\tau} \mathbb{C}$ is a CM abelian variety of type Φ as well. (It is important for later to normalize what we meant by τA here, we meant, writing A as set of points in $\mathbb{CP}^N$, say, and τA means the subset where coordinates are the original ones acted by τ.)

(2) As τA is also a CM abelian variety of type Φ. it is of similar form (but with possibly different ideal $\mathfrak{a}'$. Then there exists an E-linear isomorphism

$$\alpha : H_1(A(\mathbb{C}), \mathbb{Q}) \xrightarrow{\cong} H_1((\tau A)(\mathbb{C}), \mathbb{Q})$$

The ambiguity of this α is up to a scalar in $E^\times$.

There is another isomorphism,

$$\tau : H_1^{\mathrm{et}}(A(\mathbb{C}), \mathbb{A}_f) = \widehat{T}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\tau} \widehat{T}(\tau A) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1^{\mathrm{et}}((\tau A)(\mathbb{C}), \mathbb{A}_f).$$

This is given by applying τ to each of the coordinates of the torsion points of A.

So there exists $\tilde{f}_\Phi(\tau) \in \mathbb{A}_{E,f}^\times$ making the following diagram commute

$$\begin{array}{ccc}
H_1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{A}_f & \xrightarrow{\qquad\qquad\cong\qquad\qquad} & H_1^{\mathrm{et}}(A, \mathbb{A}_f) \\
\downarrow{\scriptstyle \alpha\otimes 1} & & \downarrow{\scriptstyle \tau} \\
H_1((\tau A)(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{A}_f \xrightarrow{\cong} H_1^{\mathrm{et}}(\tau A, \mathbb{A}_f) & \xrightarrow{\cdot \tilde{f}_\Phi(\tau)} & H_1^{\mathrm{et}}(\tau A, \mathbb{A}_f)
\end{array}$$

Convince yourself that $\tilde{f}_\Phi(\tau)$ is well-defined in $E^\times\backslash\mathbb{A}_{E,f}^\times$.

What Shimura reciprocity map says in this case is that $f_\Phi(\tau)$ is given by the Shimura reciprocity map, in the following sense. Write T for $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ show that $\mu_\Phi : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ defined by the CM type Φ is defined over the reflex field E_Φ; so that we have $\mu_\Phi : \mathbb{G}_{m,E_\Phi} \to T_{E_\Phi}$.

$$\mathrm{rec}_\Phi : \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/E_\Phi) \to \mathrm{Gal}(E_\Phi^{\mathrm{ab}}/E_\Phi) \xleftarrow{\mathrm{Art}} E_\Phi^\times\backslash\mathbb{A}_{E_\Phi,f}^\times \xrightarrow{\mu_\Phi} T(E_\Phi)\backslash T(\mathbb{A}_{E_\Phi}^\times) \xrightarrow{\mathrm{norm}} E^\times\backslash\mathbb{A}_{E,f}^\times.$$

Then $\tilde{f}_\Phi(\tau) = \mathrm{rec}_\Phi(\tau)$ in $E^\times\backslash\mathbb{A}_{E,f}^\times$.

Problem 7.6 (Geometric connected components of Shimura varieties)**.** Let (G, X) denote a Shimura datum and let G_{ab} denote the maximal abelian quotient of G and $\nu : G \to G_{\mathrm{ab}}$ the natural map. Then each $h : \mathbb{S} \to G_{\mathbb{R}}$ in X induces the same homomorphism $h_{\mathrm{ab}} : \mathbb{S} \to G_{\mathbb{R}} \to G_{\mathrm{ab},\mathbb{R}}$. So we have a natural morphism of Shimura data

$$(G, X) \to (G_{\mathrm{ab}}, \{h_{\mathrm{ab}}\}).$$

If $K \subseteq G(\mathbb{A}_f)$ is an open compact subgroup then $\nu(K)$ is an open compact subgroup of $G_{\mathrm{ab}}(\mathbb{A}_f)$.

(This problem is taken from Milne's Introduction to Shimura varieties, [Mi05, Theorem 5.17].) **Assume that the derived subgroup G^{der} is simply-connected.** Then we will prove below that the natural map $\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{\nu(K)}(G_{\mathrm{ab}}, \{h_{\mathrm{ab}}\})$ "almost" induces an isomorphism on the set of geometric connected components. More precisely, let Z denote the center of G and set

$$G_{\mathrm{ab}}(\mathbb{R})^\dagger := \mathrm{Im}(Z(\mathbb{R}) \to G_{\mathrm{ab}}(\mathbb{R})) \quad \text{and} \quad G_{\mathrm{ab}}(\mathbb{Q})^\dagger := G_{\mathrm{ab}}(\mathbb{Q}) \cap G_{\mathrm{ab}}(\mathbb{R})^\dagger.$$

Then the natural map

$$\mathrm{Sh}_K(G,X) \to G_{\mathrm{ab}}(\mathbb{Q})^\dagger \backslash G_{\mathrm{ab}}(\mathbb{A}_f)/\nu(K) \tag{7.6.1}$$

induces a bijection on the geometric connected components.

(1) First look at what this statement entails in some examples: consider $G = \mathrm{GL}_{2,\mathbb{Q}}$, that is the case of modular curves. In this case, the maximal abelian quotient is given by $\nu = \det : \mathrm{GL}_{2,\mathbb{Q}} \to G_{\mathrm{ab}} = \mathbb{G}_{m,\mathbb{Q}}$. So $G_{\mathrm{ab}}(\mathbb{R})^\dagger = \mathbb{R}^{>0}$ and $G_{\mathrm{ab}}(\mathbb{Q})^\dagger = \mathbb{Q}^{\times,>0}$. If we take $\Gamma_1(N)$-level structure, it corresponds to $\widehat{\Gamma}_1(N) = \big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}});\ c \equiv 1, d \equiv 0 \pmod{N} \big\}$. The determinant is the entire $\widehat{\mathbb{Z}}^\times$. So

$$\pi_0^{\mathrm{geom}}(\mathrm{Sh}_{\widehat{\Gamma}_1(N)}(\mathrm{GL}_{2,\mathbb{Q}})) = \mathbb{Q}^{\times,>0} \backslash \mathbb{A}_f^\times / \widehat{\mathbb{Z}}^\times = \{1\}.$$

In this case, the modular curve is always connected.

On the other hand, when the level structure is $\Gamma(N)$, corresponding to $\widehat{\Gamma}(N) := \big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}});\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \big\}$, whose determinant is $(1+N\widehat{\mathbb{Z}})^\times$. In this case

$$\pi_0^{\mathrm{geom}}(\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}})) = \mathbb{Q}^{\times,>0} \backslash \mathbb{A}_f^\times / (1+N\widehat{\mathbb{Z}})^\times = (\mathbb{Z}/N\mathbb{Z})^\times.$$

We can further discuss the Galois action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ on the set of geometric connected component (which comes from the Shimura reciprocity map for G_{ab} and $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}_{2,\mathbb{C}} \xrightarrow{\nu} \mathbb{G}_{m,\mathbb{C}}$ sending $z \to z$)

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{Art} \mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}^\times_{>0} = \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times.$$

From this, we see that the Galois action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$ on $(\mathbb{Z}/N\mathbb{Z})^\times$ is factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. There is another way to explain this: $\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}})$ is an irreducible curve over $\mathbb{Q}(\zeta_N)$, but when we view it naturally over $\mathbb{Q}$ instead, and make base change, we see that $\mathrm{Sh}_{\widehat{\Gamma}(N)}(\mathrm{GL}_{2,\mathbb{Q}}) \times_{\mathbb{Q}} \mathbb{C}$ has $(\mathbb{Z}/N\mathbb{Z})^\times$-geometric connected components.

(2) Now we indicate the proof of (7.6.1). For this, we need to accept a few blackbox theorems from [PR94, Theorem 6.4, 6.6]: (these are very useful statements)

- (vanishing of nonarchimedean cohomology for simply-connected groups) If G is simply-connected semisimple group over $\mathbb{Q}_\ell$, then $H^1(\mathbb{Q}_\ell, G) = \{1\}$.
- (Hasse principle for simply-connected group and adjoint group) For an algebraic group G over $\mathbb{Q}$, we define

  $$Ш^1_f(\mathbb{Q}, G) := \mathrm{Ker}\big(H^1(\mathbb{Q}, G) \to \prod_{\ell \neq \infty} H^1(\mathbb{Q}_\ell, G)\big).$$

  Then if G is simply-connected and semisimple, then

  $$Ш^1_f(\mathbb{Q}, G) \to H^1(\mathbb{R}, G)$$

is an isomorphism. (If G is semisimple and adjoint, this is injective.)

- If G is a simply-connected real reductive group (or a compact real reductive group), then $G(\mathbb{R})$ is connected.
- (Strong approximation for simply-connected groups) If G is a simply-connected group over a number field F; suppose that v is a place of F such that $G(F_v)$ is non-compact at each F-simple factor of G, then $G(F)$ is dense in $G(\mathbb{A}_F^{(v)})$.

Applying these statements, we prove the following in turns.

- Let X^+ denote the connected component of X, then the stabilizer of X^+ under the $G(\mathbb{R})$ action is $G(\mathbb{R})_+ :=$ preimage of the connected component of $G_{\mathrm{ad}}(\mathbb{R})$ in $G(\mathbb{R})$. Set $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Then
$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f)/K.$$
- If G^{der} is simply-connected, then $G(\mathbb{R})_+ = G^{\mathrm{der}}(\mathbb{R}) \cdot Z(\mathbb{R})$.
- If G^{der} is simply-connected, then $G(\mathbb{A}_f) \to T_{\mathrm{ab}}(\mathbb{A}_f)$ is surjective and sends open compact subgroups to open compact subgroups.

Concludes eventually that (7.6.1) induces an isomorphism between geometric connected components.

Remark: the geometric connected component of more general Shimura varieties is somewhat subtle, see the discussion in Deligne's article in Corvallis.

References

[Mi05] J. Milne, Introduction to Shimura varieties, Harmonic Analysis, the Trace Formula and Shimura Varieties, Clay Mathematics Proceedings, Volume 4 (2005), 265-378.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, Vol. **139**, Academic Press Inc. Boston, MA, 1994, xii+614 pp.

Hints. Problem 7.1: The converse map is given by taking the Galois descent from $\mathbb{C}$ to $\mathbb{R}$ of the homomorphism

$$\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu,\, c\circ\mu\circ c} T_{\mathbb{C}}$$

Problem **??**: