Exercise for Talk 8: Dual BGG complex and Hodge theory.

Introduce automorphic local systems, vector bundles, and canonical P-torsors on a Shimura varieties. Following Faltings' paper to discuss dual BGG complex construction and the spectral sequence.

Problem 8.1 (Explicit computation of Verma modules for $\mathfrak{sl}_2$)**.** We explicitly construct the structure of Verma module for $\mathfrak{g} = \mathfrak{sl}_2$ (over $\mathbb{C}$), namely the trace zero 2-by-2 matrices. We take the Borel subgroup $\mathfrak{b}$ to consist of the upper triangular matrices, and the Cartan subalgebra to be the diagonal matrices $\mathfrak{h}$. Write explicitly the following matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For an algebraic representation V, we write $V_n = \{v \in V;\ H(v) = nv\}$.

(1) Verify that $E : V_n \to V_{n+2}$ and $F : V_n \to V_{n-2}$.

(2) For $n \in \mathbb{R}$, write $\mathbb{C}_n$ for the one-dimensional representation of $\mathfrak{h}$, with basis e such that $He = ne$. Consider the *Verma module* $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$, which admits a basis given by $F^r \otimes e$, which is an eigenvector of H of eigenvalue $n - 2r$. Show that (for $r \geq 0$)

$$E(F^r \otimes e) = r(n - r + 1) \cdot F^{r-1} \otimes e$$

(3) From this explicit description, deduce that $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$ is irreducible if and only if $n \notin \mathbb{Z}_{\geq 0}$. If $n \geq 0$, there is a natural exact sequence

$$0 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-n-2} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_n \to \mathrm{Sym}^n \mathbb{C}^2 \to 0,$$

where $\mathbb{C}^2$ carries the standard representation of $\mathfrak{sl}_2$.

Problem 8.2 (Casimir operator)**.** Consider the three operators in $\mathfrak{sl}_2$:

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We explain a general way to construct Casimir operator (for semisimple Lie algebras).

(1) Consider the Killing form (which is symmetric bilinear) defined on $\mathfrak{sl}_2$:

$$\langle \cdot, \cdot \rangle : \quad \begin{aligned} \mathfrak{sl}_2 \times \mathfrak{sl}_2 &\longrightarrow \mathbb{C} \\ (X, Y) &\longmapsto \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \in \mathbb{C}. \end{aligned}$$

Show that, with respect to the basis $\{F, H, E\}$, the matrix for the symmetric bilinear Killing form is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

From this, we see that the dual basis are $\{\frac{1}{4}E, \frac{1}{8}H, \frac{1}{4}F\}$ in order.

(2) Prove abstractly that the Killing form is G-equivariant, i.e. $\langle \mathrm{ad}_g(X), \mathrm{ad}_g(Y) \rangle = \langle X, Y \rangle$, for $X, Y \in \mathfrak{sl}_2$ and $g \in \mathfrak{sl}_2$. From this, deduce purely abstractly that

$$C := E \cdot E^* + F \cdot F^* + H \cdot H^* = \frac{1}{4}(EF + FE + \frac{1}{2}H^2)$$

commutes with $\mathfrak{sl}_2$ in $U(\mathfrak{sl}_2)$, namely C belongs to the center $Z(\mathfrak{sl}_2)$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$. (Note that: this abstract construction works for every semisimple Lie algebra $\mathfrak{g}$, producing a *Casimir* operator of degree 2 in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. In the case of $\mathfrak{sl}_2$, one can show that $Z(\mathfrak{sl}_2) = \mathbb{C}[C]$ is the polynomial algebra generated by this degree 2 Casimir operator. For general semisimple algebra $\mathfrak{g}$, the generators of $Z(\mathfrak{g})$ may of higher degree.)

Remark on notation: In different literature, the definition Casimir operator may be differed by a scalar. This is related to the normalization of Killing form we are using.

(3) Consider the Verma module $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$ in the previous question. Show that C acts on this Verma module by multiplication by $\frac{1}{8}(n^2 + 2n)$.

In particular, this shows that the C-action on $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$ and $U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-n-2}$ are the same.

Problem 8.3 (Weyl character formula in terms of Verma modules)**.** We now explain how to deduce Weyl character formula for highest weight representations from the BGG resolution.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{h}$ a Cartan subalgebra. All the roots of $\mathfrak{g}$ is denoted by the subset $\Phi \subset \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, that is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

in which $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ for the set Φ^+ of positive roots. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ denote the half sum of all positive roots. There exists a subset $\Delta \subset \Phi^+$, called the set of simple roots, such that every root $\alpha \in \Phi^+$ is a unique $\mathbb{Z}_{\geq 0}$-linear combination of simple roots.

Recall that the Weyl group W acts on $\mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, generated by reflections s_α along each simple root $\alpha \in \Delta$ (with respect to the natural Killing form on $\mathfrak{g}$ given by $(X, Y) \mapsto \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y; \mathfrak{g})$). This Weyl group preserves the subset Φ. Each $w \in W$ can be written as the product of s_α's with $\alpha \in \Delta$. The minimal length of such expression is called the length of w, denoted by $\ell(w)$.

Let V be a representation of $\mathfrak{g}$ on which $\mathfrak{h}$ acts semisimply, that is we have a decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V(\lambda).$$

A weight of V is a $\lambda \in \mathfrak{h}^*$ for which $V(\lambda) \neq 0$. For such a representation, we define the character to be the power series

$$\mathrm{Char}(V) := \sum_{\lambda \in \mathfrak{h}^*} \dim V(\lambda) \cdot e^\lambda \ \in \ \mathbb{C}\big[e^\lambda, \lambda \in \mathfrak{h}^*\big] / \big(e^\lambda \dot{e}^\mu - e^{\lambda+\mu}\big).$$

(1) By Poincaré–Birkhoff–Witt theorem the natural map $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \to U(\mathfrak{g})$ is an isomorphism of vector spaces, where $\mathfrak{n}^-$ is the opposite nilpotent Lie algebra. In addition, for each simple root $\alpha \in \Delta$, fix a basis f_α of $\mathfrak{g}_{-\alpha}$. Then $U(\mathfrak{n}^-)$ has a $\mathbb{C}$-basis given by

$$\Big\{ \prod_{\alpha \in \Delta} n_\alpha^{i_\alpha};\ i_\alpha \in \mathbb{Z} \Big\}.$$

For $\lambda \in \mathfrak{h}^*$, consider the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ (where $\mathfrak{b}$ acts on $\mathbb{C}_\lambda$ through the quotient $\mathfrak{h}$ and character λ), show that

$$\mathrm{Char}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda\big) = e^\lambda \prod_{\alpha \in \Delta} \big(1 + e^\alpha + e^{2\alpha} + \cdots\big) = \frac{e^\lambda}{\prod_{\alpha \in \Delta}(1 - e^\alpha)}$$

(2) There is a slightly twisted action of W on $\mathfrak{h}^*$ given by

$$w \star \lambda = w(\lambda + \rho) - \rho,$$

i.e. it is the usual Weyl group action but center shifted to the point $-\rho$.

Now the usual BGG resolution says that the highest weight representation V_λ of $\mathfrak{g}$ with highest weight λ (such that $\langle \lambda, \check{\alpha}\rangle \geq 0$ for every simple coroot $\check{\alpha}$), admits the following resolution.

$$\cdots \to \bigoplus_{\substack{w\in W\\ \ell(w)=i}} U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_{w\star\lambda}\cdots \to \bigoplus_{\alpha\in\Delta} U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_{s_\alpha\star\lambda}\to U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_\lambda \to V_\lambda \to 0$$

Using this to show that

$$\begin{aligned}\mathrm{Char}(V_\lambda) &= \frac{e^\lambda}{\prod_{\alpha\in\Delta}(1-e^\alpha)} - \sum_{\alpha\in\Delta}\frac{e^{s_\alpha\star\lambda}}{\prod_{\alpha\in\Delta}(1-e^\alpha)} + \cdots + (-1)^\ell \sum_{\substack{w\in W\\ \ell(w)=i}}\frac{e^{w\star\lambda}}{\prod_{\alpha\in\Delta}(1-e^\alpha)}+\cdots\\ &= \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w\star\lambda}}{\prod_{\alpha\in\Delta}(1-e^\alpha)}.\end{aligned}$$

Problem 8.4 (Dual BGG resolution for $GU(1,2)$ Shimura varieties)**.** Let $E=\mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field and V a Hermitian space over E of signature $(1,2)$ at infinity. Let $G=GU(V)$ denote the unitary group of V with similitudes in $\mathbb{G}_m$, which is an algebraic group over $\mathbb{Q}$. (Fix a choice of $\sqrt{d}$ in $\mathbb{C}$.) Consider

$$\begin{aligned} h: \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times &\longrightarrow G(\mathbb{R}) \subseteq \mathrm{GL}_3(\mathbb{C})\\ z &\longmapsto \begin{pmatrix} z & & \\ & \bar{z} & \\ & & \bar{z}\end{pmatrix}.\end{aligned}$$

Fix a sufficiently small level structure $K_f \subset G(\mathbb{A}_f)$. Write $\tau: E\to\mathbb{C}$ for the embedding so that $\tau(\sqrt{d})\in\mathbb{R}_{>0}i$. The moduli space of abelian varieties for G sends a E-scheme S to the set of quasi-isogenous classes of triples (A,λ,η), where

- A is an abelian variety over S with an $\mathcal{O}_E$-action of dimension 3 such that $\omega_{A^\vee/S}$ decomposes as the direct sum $\omega_{A^\vee/S,\tau}\oplus\omega_{A^\vee/S,\bar{\tau}}$ of locally free sheaves of rank 1 and 2, respectively, where $\mathcal{O}_E$ acts on each factor via the scalar multiplication or the scalar multiplication by conjugates, respectively;
- $\lambda: A\to A^\vee$ is a quasi-isogeny such that $n\lambda$ is a polarization for some $n\in\mathbb{N}$ such that the Rosati involution induces complex conjugation on A;
- η is the level structure (which we do not further discuss here).

The picture at infinite is as follows: $K_\infty = G(U(1)\times U(2))$. $\mathfrak{g}_\mathbb{C} = \mathbb{C}\oplus\mathfrak{gl}_3$ and $\mathfrak{k}_\mathbb{C} = \mathbb{C}\oplus\mathfrak{gl}_1\oplus\mathfrak{gl}_2$. We choose the Cartan subalgebra to be $\mathfrak{h}=\mathbb{C}\oplus\mathfrak{gl}_1^{\oplus 3}$ and the Borel subalgebra to be the "upper triangular" one. In this case, $\mathfrak{p}^+ = \left(\begin{smallmatrix}0&*&*\\0&0&0\\0&0&0\end{smallmatrix}\right)$ and $\mathfrak{p}^- = \left(\begin{smallmatrix}0&0&0\\ *&0&0\\ *&0&0\end{smallmatrix}\right)$; so $\mathfrak{q}=\mathbb{C}\oplus\left(\begin{smallmatrix}*&*&*\\0&*&*\\0&*&*\end{smallmatrix}\right)$. Over the whole moduli space, we then have vector bundles

$$\omega_\tau = \omega_{\mathcal{A}^\vee/X,\tau}\subset \mathcal{H}^{\mathrm{dR}}_{1,\tau} = \mathcal{H}^{\mathrm{dR}}_1(\mathcal{A}/X)_\tau \quad\text{and}\quad \omega_{\bar{\tau}}\subset\mathcal{H}^{\mathrm{dR}}_{1,\bar{\tau}}.$$

with ranks $\mathrm{rank}\,\mathcal{H}^{\mathrm{dR}}_{1,\tau} = \mathrm{rank}\,\mathcal{H}_{1,\bar{\tau}} = 3$, $\mathrm{rank}\,\omega_\tau = 1$ and $\mathrm{rank}\,\omega_{\bar{\tau}}=2$.

The weight of $\mathfrak{g}$ (namely the characters of $\mathfrak{h}$) can be labeled by $\mathbb{Z}\times\mathbb{Z}^3$ for each of the factors on $\mathfrak{h}$. The tuple $(w;k_1,k_2,k_3)$ is dominant if $k_1\geq k_2\geq k_3$ and is K_∞-dominant if $k_2\geq k_3$. In this case, it corresponds to the automorphic vector bundle

$$\omega^{(w,k_1,k_2,k_3)} := \omega^{k_1}_{A^\vee,\tau}\otimes \mathrm{Sym}^{k_2-k_3}(\omega_{A^\vee,\bar{\tau}})\otimes\big(\wedge^2\omega_{A^\vee,\bar{\tau}}\big)^{w-k_2}\otimes\big(\wedge^3\mathcal{H}^{\mathrm{dR}}_{1,\tau}\big)^w.$$

(Basically, one can show that $\wedge^3\mathcal{H}_{1,\tau}^{\mathrm{dR}}$ has trivial Chern class.) The essential main player are the first three terms.

(1) In this case, the Weyl group $W = S_3$ and W_c is the subgroup generated by the permutation (23). Give the group W_{nc}.

(2) Compute the ρ in this case.

(3) Let $\pi : \mathcal{A} \to X(\mathbb{C})$ denote the universal abelian variety; the pushforward $R^1\pi_*\underline{\mathbb{C}}_{\mathcal{A}}$ decomposes as the direct sum of $\mathcal{E}_\tau$ and $\mathcal{E}_{\bar{\tau}}$, locally constant sheaf each of rank 3 over X. (Polarization ensures that $\mathcal{E}_{\bar{\tau}} \cong \mathcal{E}_\tau^*(1)$.) Let $\mathcal{E}_\tau^{\mathrm{GL}_3}$ denote the locally constant $\mathrm{GL}_3(\mathbb{C})$-torsor defined by $\mathcal{E}_\tau$. Let $(w; k_1, k_2, k_3)$ be a dominant weight. Let $V(\underline{k})$ denote the highest weight representation of GL_3 with highest weight (k_1, k_2, k_3). Then we have a locally constant sheaf

$$\mathcal{E}^{(w;k_1,k_2,k_3)} := \left(\mathcal{E}_\tau^{\mathrm{GL}_3} \times^{\mathrm{GL}_3(\mathbb{C})} V(\underline{k})\right) \otimes \wedge^3(\mathcal{E}_\tau).$$

Using the recipe for dual BGG resolution, give explicit resolution in terms of the moduli problem above.

Hints. Problem 8.1: (3) To show irreducibility, we start with any vector in the given representation of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$, we may apply the semisimple H-action to ensure that this vector v is an eigenvectore, which up to scalar must be one of $F^r \otimes e$. While we apply weight raising operator E to $F^r \otimes e$, we see that we will always get a nonzero multiple of $F^{r-1} \otimes e$ (and eventually to e), except when $n = r$, this is where the irreducibility of general $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_n$ fails. The BGG resolution in this case follows from this.

Problem 8.4: (1) $W_{nc} = \{1,\ (12),\ (132)\}$ with length 0, 1, and 2, respectively.

(2) All positive roots are $(0; 1, -1, 0)$, $(0; 1, 0, -1)$, $(0; 0, 1, -1)$. Thus $\rho = (0; 1, 0, -1)$.

(3) The dual of $V(w; k_1, k_2, k_3)$ is $V(-w; -k_3, -k_2, -k_1)$. In terms of BGG resolution, we are using

$$0 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_{(-w;-k_1-2;-k_3+1,-k_2+1)} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_{(-w;-k_2-1,-k_3+1,-k_1)} \to$$
$$U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_{(-w;-k_3,-k_2,-k_1)} \to V(w; k_1, k_2, k_3)^* \to 0$$

where $V(w; k_1, k_2, k_3)$ is the irreducible $\mathfrak{g}$-representation with highest weight $(w; k_1, k_2, k_3)$, and $W(w; k_1, k_2, k_3)$ is the irreducible $\mathfrak{q}$-representation with highest weight $(w; k_1, k_2, k_3)$. Reverting this back to sheaves, we get

$$0 \to \mathcal{E}^{(w;k_1,k_2,k_3)} \to \omega^{(w;k_3,k_1,k_2)} \to \omega^{(-w;k_2+1,k_3-1,k_1)} \to \omega^{(w;k_1+2,k_2-1,k_3-1)} \to 0$$