Introduction to modular curves and Siegel modular varieties

§1. Adelic description of modular curves

Let $N \geq 4$, consider $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ N|c, N|d-1 \right\}$

Then the modular curve is $Y_1(N)(\mathbb{C}) := \Gamma_1(N) \backslash \mathfrak{h}$ where $\mathfrak{h} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

Let $\mathbb{A}_f :=$ finite adeles of $\mathbb{Q}$.

Theorem 1 There is an isomorphism $\Gamma_1(N) \backslash \mathfrak{h} \xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / \widehat{\Gamma_1(N)}$

where $\mathfrak{h}^{\pm} := \mathbb{C} \setminus \mathbb{R}$. $\widehat{\Gamma_1(N)} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) \mid c, d-1 \in N\hat{\mathbb{Z}} \right)$, $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

Need a black box: Strong Approximation: $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_f)$

(In general, if G is a simply-connected simple group over a number field F,

and v is a place of F s.t. $G(F_v)$ is not compact,

then $G(F)$ is dense in $G(\mathbb{A}_F^{(v)})$ ← adeles away from v.

E.g. $G = SL_n, Sp_{2n}, D^{Nm=1}$, or $SU(V)$

(D: division alg over F; V hermitian space for E/F)

Cor: $SL_2(\mathbb{A}_f) \subseteq SL_2(\mathbb{Q}) \cdot \widehat{\Gamma_1(N)}$

$GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q}) \cdot \widehat{\Gamma_1(N)}$

off by $\mathbb{A}_f^\times$ but $\mathbb{A}_f^\times = \mathbb{Q}^\times \cdot \hat{\mathbb{Z}}^\times$, can first modify an element in $GL_2(\mathbb{A}_f)$ into $SL_2(\mathbb{A}_f)$

Remark: ↑ This argument need the class group of $\mathbb{Z}$ to be trivial

Proof of Theorem 1: Consider $GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / \widehat{\Gamma_1(N)}$

By Cor, every coset can be represented by $(z, 1) \in \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)$

The ambiguity lies in $\widetilde{\Gamma_1(N)} := GL_2(\mathbb{Q}) \cap \widehat{\Gamma_1(N)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \mid N | c-1, d \right\}$

So that the above double coset is $\widetilde{\Gamma_1(N)} \backslash \mathfrak{h}^{\pm}$

$SL_2(\mathbb{Z})$ has index 2 in $GL_2(\mathbb{Z})$ as $\det(GL_2(\mathbb{Z})) = \mathbb{Z}^\times = \{\pm 1\}$

$$\Rightarrow \widetilde{\Gamma_1(N)}\backslash \mathfrak{h}^{\pm} = \Gamma_1(N)\backslash \mathfrak{h}.$$

§2 Moduli interpretations

Recall: An elliptic curve E over $\mathbb{C}$ takes the form of $E(\mathbb{C}) = (\mathbb{C},+)/\mathbb{Z}\oplus\mathbb{Z}\tau$.

But $\tau \in \mathfrak{h}$ is uniquely determined up to the action of $SL_2(\mathbb{Z})$.

Theorem 2 Assume $N \geq 4$. $Y_1(N)$ is the moduli space of elliptic curves with an N-torsion point, that is, there exists $\mathcal{E}_{univ}$ = universal elliptic curve,

not important here, but will see μ_N instead for many reference.

$$\text{zero section} \rightarrow s \left(\downarrow \pi \right. \quad \text{together with } i_{univ} : (\mathbb{Z}/N\mathbb{Z})_Y \hookrightarrow \mathcal{E}[N]$$

$$Y_1(N) = Y \qquad \text{an embedding of group scheme}$$

s.t. for every $\mathbb{C}$-scheme S together with an elliptic curve E/S & an embedding $i : (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$ there exists a unique morphism $\alpha : S \to Y$ s.t. $E = \alpha^* \mathcal{E}_{univ}$ and $i = \alpha^* i_{univ}$

Proof: Over $\tau \in \mathfrak{h}$, we define $E_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$, $i_{univ} : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E_\tau[N]$

$$1 \longmapsto \frac{1}{N}$$

Then $(E_\tau)_{\tau\in\mathfrak{h}}$ varies holomorphically as τ moves.

Then $\mathcal{E}$ = quotient of the family by the $\Gamma_1(N)$-action. □.

Will give a slightly different approach to get the adelic parametrization.

* $\hat{T}(E) :=$ Tate module of $E = \varprojlim_n E[n]$,

 (If E is over a $\mathbb{Q}$-scheme S, $\hat{T}(E)$ is an étale $\hat{\mathbb{Z}}$-sheaf of rank 2.)

Claim: If $E/\mathbb{C}$ is an elliptic curve, giving an embedding $i : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$ is equivalent to a $\widehat{\Gamma_1(N)}$-orbit of isomorphisms $\hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E)$

$$\circlearrowleft \widehat{\Gamma_1(N)} \text{ (acting on } \hat{\mathbb{Z}}^{\oplus 2})$$

Proof: Note: $\hat{T}(E) \twoheadrightarrow E[N]$. Stabilizer of i is precisely $\widehat{\Gamma_1(N)}$.

Remark: (Language issue) If E is over a local noetherian $\mathbb{Q}$-scheme S, we will need to take a $\pi_1(S,s)$-stable $\widehat{\Gamma_1(N)}$-orbit of isomorphisms $\hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E)$.

Or in a fancier language, $\underline{\mathrm{Isom}}(\hat{\mathbb{Z}}^{\oplus 2}, \hat{T}(E))$ is an étale $GL_2(\hat{\mathbb{Z}})$-torsor, a level-N-structure is a section of $\underline{\mathrm{Isom}}(\hat{\mathbb{Z}}^{\oplus 2}, \hat{T}(E))/\widehat{\Gamma_1(N)}$.

<u>Another proof of Theorem 1:</u>

Let (E, i) be an elliptic curve over $\mathbb{C}$ with an embedding $i : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$.

As above, i corresponds to a $\widehat{\Gamma_1(N)}$-orbit of isomorphisms $\eta : \hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E)$.

- The elliptic curve $E/\mathbb{C}$ has three features:
  - <u>Betti homology</u>: $H_1(E(\mathbb{C}), \mathbb{Z})$
  - <u>Étale homology</u> $H_1^{et}(E, \hat{\mathbb{Z}}) \simeq \hat{T}(E)$

) <u>comparison</u>: $H_1(E(\mathbb{C}),\mathbb{Z}) \otimes \hat{\mathbb{Z}} \cong H_1^{et}(E, \hat{\mathbb{Z}})$

  - <u>de Rham filtration</u>: $0 \to H^0(E, \Omega^1_E) \to H^1_{dR}(E/\mathbb{C}) \longrightarrow H^1(E, \mathcal{O}_E) \to 0$

  $\overset{\text{dualization}}{\rightsquigarrow}$ $0 \to \omega_{E^\vee/\mathbb{C}} \to H_1^{dR}(E/\mathbb{C}) \longrightarrow \mathrm{Lie}_{E/\mathbb{C}} \to 0$

  where $\omega_{E^\vee/\mathbb{C}} = H^0(E^\vee, \Omega^1_{E^\vee})$, $E^\vee = E$, and $H_1^{dR}(E/\mathbb{C}) \cong H^1_{dR}(E^\vee/\mathbb{C})$

) <u>Comparison</u>: $H_1(E(\mathbb{C}),\mathbb{Z}) \otimes \mathbb{C} \simeq H_1^{dR}(E/\mathbb{C})$

<u>Theorem 2</u>. There is a bijection $Y_1(N)(\mathbb{C}) \simeq GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)/\widehat{\Gamma_1(N)}$.

Proof: Starting with (E, η). We choose an isomorphism $\beta : H_1(E(\mathbb{C}), \mathbb{Q}) \simeq \mathbb{Q}^{\oplus 2}$.

Then we can construct elements in $GL_2(\mathbb{A}_f)$ and $\mathfrak{h}^{\pm}$ as follows:

$$* \quad \mathbb{A}_f^{\oplus 2} = \hat{\mathbb{Z}}^{\oplus 2} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\simeq]{\eta} H_1^{et}(E, \hat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\sim]{\text{Comparison}} H_1(E(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{A}_f) \xrightarrow[\sim]{\beta} \mathbb{A}_f^{\oplus 2}$$

$\hookrightarrow$ gives an element $g_f \in GL_2(\mathbb{A}_f)$

$$* \quad \omega_{E^\vee/\mathbb{C}} \subseteq H_1^{dR}(E/\mathbb{C}) \xrightarrow[\sim]{\text{comparison}} H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \xrightarrow[\simeq]{\beta} \mathbb{C}^{\oplus 2}$$

$\hookrightarrow$ gives an element $\tau \in \mathbb{P}^1(\mathbb{C})$ (can prove that it does not belong to $\mathbb{R}$)

So, we get $(E, \eta) \rightsquigarrow (\tau, g_f) \in \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)$.

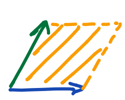
This association depends on
- * choice of η in the $\widehat{\Gamma_1(N)}$-orbit $\rightsquigarrow$ g_f mod $\widehat{\Gamma_1(N)}$
- * choice of isom. β, if $\beta' = h \circ \beta$, then

$$(g_f', \tau') = (h \cdot g_f, h \cdot \tau)$$

Putting these together gives a map $Y_1(N)(\mathbb{C}) \to GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)/\widehat{\Gamma_1(N)}$.

Conversely, given $(\tau, g_f) \in \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)$,

we can define $E = E_\tau = \mathbb{C}/\mathbb{C} \oplus \mathbb{C}\tau$,



there's a somewhat natural isom $\beta : H_1(E_\tau, \mathbb{Q}) \simeq \mathbb{Q}^{\oplus 2}$

$\rightarrow \;\leftrightarrow (1,0)$

$\uparrow \;\leftrightarrow (0,1)$

Ex: check that the filtration corresponds to τ.

- g_f gives a natural isom. $\mathbb{A}_f^{\oplus 2} \simeq H_1^{et}(E_\tau, \mathbb{A}_f)$ by inverting the argument above

$$\begin{array}{ccc} \mathbb{A}_f^{\oplus 2} & \simeq & H_1^{et}(E_\tau, \mathbb{A}_f) \\ \cup\!| & & \cup\!| \\ \hat{\mathbb{Z}}^{\oplus 2} & \leftrightarrow & H_1^{et}(E_\tau, \hat{\mathbb{Z}}) \end{array}$$

$\uparrow$ do not match.

Say $M_1 \hat{\mathbb{Z}}^{\oplus 2} \subseteq H_1^{et}(E_\tau, \hat{\mathbb{Z}}) \subseteq \frac{1}{M_2} \hat{\mathbb{Z}}^{\oplus 2}$

Consider $\underbrace{\dfrac{M_1 \hat{\mathbb{Z}}^{\oplus 2}}{M_1 M_2 H_1^{et}(E_\tau, \hat{\mathbb{Z}})}}_{=:\, G} \subseteq \dfrac{H_1^{et}(E_\tau, \hat{\mathbb{Z}})}{M_1 M_2 \cdot H_1^{et}(E_\tau, \hat{\mathbb{Z}})} \cong E_\tau[M_1 M_2]$

G is a subgroup of E_τ

Set $E := E_\tau / G$, then the quasi-isogeny

$$E \xrightarrow{\times M_2} E \longleftarrow E_\tau$$

induces $H_1^{et}(E, \hat{\mathbb{Z}}) \xrightarrow{\times M_2} H_1^{et}(E, \hat{\mathbb{Z}}) \longleftarrow H_1^{et}(E_\tau, \hat{\mathbb{Z}})$

$$\eta^{-1} \searrow\!\sim \qquad \hat{\mathbb{Z}}^{\oplus 2} \subseteq \mathbb{A}_f^{\oplus 2} \overset{g_f}{\simeq} H_1^{et}(E_\tau, \mathbb{A}_f) \quad (\cap\!| \text{ from } H_1^{et}(E_\tau, \hat{\mathbb{Z}}))$$

(E, η) is what we needed. $\square$.

§3 Moduli problems over $\mathbb{Q}$ and $\mathbb{Z}_{(p)}$

Black Box Theorem. Assume $N \geq 4$.

The functor $\mathcal{M}_K : Sch_{/\mathbb{Q}}^{\text{loc. noe.}} \longrightarrow \mathcal{S}ets$

$S \longmapsto \mathcal{M}(S) = \left\{ \begin{array}{l} \text{isom. classes of } (E, \eta) : \ * \ E \text{ elliptic curve} / S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta : \hat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\sim} \hat{T}(E) \text{ is a } \pi_1(S, \bar{s})\text{-stable } \Gamma_1(N)\text{-orbit of isomorphisms} \end{array} \right\}$

is representable by a (geometrically connected) smooth curve $Y_1(N)$ over $\mathbb{Q}$.

Remark. This is equivalent to the earlier moduli problem, but we can easily modify $\widehat{\Gamma_1(N)}$ to any open compact subgroup $K \subseteq GL_2(\hat{\mathbb{Z}})$. **Exercise**: Reformulate the moduli problem for $K \subseteq GL_2(\mathbb{A}_f)$.

But one needs to be careful when talking about integral versions.

Let p be a prime number.

$\mathbb{Z}_{(p)}$ = localization at (p) ; $\hat{\mathbb{Z}}^{(p)} := \prod_{\ell \neq p} \mathbb{Z}_\ell$ & $\mathbb{A}_f^{(p)} := \hat{\mathbb{Z}}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Q}$

Let $K^p \subseteq GL_2(\hat{\mathbb{Z}}^{(p)})$ be an open compact subgroup.

$$K := GL_2(\mathbb{Z}_p) \cdot K^p \quad (\text{so no level @ } p).$$

Can define a functor $\mathcal{M}_{K^p} : \mathrm{Sch}^{\text{loc. noe.}}_{/\mathbb{Z}_{(p)}} \longrightarrow \mathrm{Sets}$

$$S \longmapsto \mathcal{M}(S) = \left\{ \begin{array}{l} \text{isom. classes of } (E, \eta) : \ * \ E \text{ elliptic curve} / S \\ * \text{ On each connected component of } S, \text{ fixing a geometric point } \bar{s} \\ \quad \eta : \hat{\mathbb{Z}}^{(p)\oplus 2} \xrightarrow{\sim} \hat{T}(E)^{(p)} \text{ is a } \pi_1(S, \bar{s})\text{-stable } K^p\text{-orbit of isoms} \end{array} \right\}$$

It is represented by a scheme smooth over $\mathbb{Z}_{(p)}$.

§4 Siegel moduli space

Fix $V = \mathbb{Z}^{\oplus 2g} = \bigoplus_{i=1}^{2g} \mathbb{Z} e_i$, together with a symplectic form

$$\{-,-\} : V \otimes V \longrightarrow \mathbb{Z}, \quad \text{given by matrix } \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

$G := GSp(V) = GSp_{2g}$ is the group s.t. for every $\mathbb{Z}$-algebra S,

$$GSp_{2g}(S) = \{ h \in GL_{2g}(S),\ c \in S^\times \mid \{hx, hy\} = c\{x, y\} \text{ for every } x, y \in V \otimes_{\mathbb{Z}} S \}$$

$$1 \longrightarrow Sp_{2g} \longrightarrow GSp_{2g} \xrightarrow{c} \mathbb{G}_m \longrightarrow 1.$$

Fix an open compact subgroup $K \subseteq GSp_{2g}(\hat{\mathbb{Z}})$.

Theorem (Mumford) For K sufficiently small, the following functor

$$\mathcal{M}_K : \mathrm{Sch}^{\text{loc. noe.}}_{/\mathbb{Q}} \longrightarrow \mathrm{Sets}$$

$$S \longmapsto \mathcal{M}_K(S) = \left\{ \begin{array}{l} (A, \lambda, \eta) : A \text{ is an abelian variety over } S \\ \lambda : A \xrightarrow{\sim} A^\vee \text{ is a principal polarization} \\ \text{choosing a geom. point } \bar{s} \text{ on each conn. comp. of } S \end{array} \right.$$

$\eta : V_{\hat{\mathbb{Z}}} \xrightarrow{\sim} \hat{T}(A)$ is a $\pi_1(S,\bar{s})$-stable K-orbit of isoms

s.t.

$$\begin{array}{ccc} V_{\hat{\mathbb{Z}}} \times V_{\hat{\mathbb{Z}}} & \longrightarrow & \hat{\mathbb{Z}} \\ {\scriptstyle \eta}\downarrow\wr \quad {\scriptstyle \eta}\downarrow\wr & \circlearrowright & \downarrow {\scriptstyle \times c} \\ \hat{T}(A) \times \hat{T}(A) & \xrightarrow[\lambda]{} & \hat{\mathbb{Z}}(1) \end{array}$$

for some $c \in \hat{\mathbb{Z}}^\times$

is representable by a smooth variety $Sh_K(G)$ over $\mathbb{Q}$.

- Now, we explain the terms.

Every abelian variety A has a dual abelian variety $A^\vee$.

Given a polarization $\lambda : A \xrightarrow{\sim} A^\vee$, we get

$$A[n] \times A[n] \xrightarrow{\lambda} A[n] \times A^\vee[n] \xrightarrow{\text{Weil pairing}} \mu_n \quad \text{bilinear symplectic}$$

Taking the inverse limit $\Rightarrow \hat{T}(A) \times \hat{T}(A) \longrightarrow \varprojlim_n \mu_n = \hat{\mathbb{Z}}(1)$

but $\hat{\mathbb{Z}}(1)$ is <u>non-canonically</u> isomorphic to $\hat{\mathbb{Z}}$, hence the similitude factor c.

<u>Theorem</u> Let $\mathfrak{h}_g^\pm = \{ Z \in Sym_n(\mathbb{C}) : Im(Z) > 0 \text{ or } Im(Z) < 0 \}$

($>$ means totally positive.)

Then $Sh_K(G)(\mathbb{C}) \cong G(\mathbb{Q}) \backslash \mathfrak{h}_g^\pm \times G(\mathbb{A}_f) / K$.

Proof: As before, each $(A, \lambda, \eta) \in Sh_K(G)(\mathbb{C})$ gives

- Betti homology $H_1(A(\mathbb{C}), \mathbb{Z})$
- Etale homology $H_1^{et}(A(\mathbb{C}), \hat{\mathbb{Z}}) \simeq \hat{T}(A)$
- Hodge filtration $0 \to \omega_{A^\vee/\mathbb{C}} \to H_1^{dR}(A/\mathbb{C}) \to Lie_{A/\mathbb{C}} \to 0$

$$H_1^{dR}(A/\mathbb{C}) \cong H_1(A(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

<u>Mumford's AV book</u>: If we write $A \simeq \mathbb{C}^g / \Lambda$ for a lattice, then

$H_1(A(\mathbb{C}), \mathbb{Z}) \cong \Lambda$. & $\Lambda \otimes \mathbb{R} \simeq \mathbb{C}^g$ has a natural complex structure $J : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}$ (multiplication by i, so $J^2 = -1$.)

Giving a polarization λ of A amounts to giving a <u>Riemann form</u>

i.e. an alternating form $\lambda : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$

s.t. $\lambda_{\mathbb{R}}(Ju, Jv) = \lambda_{\mathbb{R}}(u,v)$ (so J belongs to Sp_{2g} as opposed to GL_{2g}) and $\lambda_{\mathbb{R}}(u, Ju) > 0$ for $u \neq 0$

Principal polarization $\Leftrightarrow \lambda$ is perfect (inducing $\Lambda \simeq Hom(\Lambda, \mathbb{Z})$)

* To get the complex uniformization, choose an isom. (up to ± 1.)

$$\alpha : (H_1(A(\mathbb{C}),\mathbb{Z}), \lambda) \simeq (V, \psi) \quad \text{(compatible with the symplectic forms)}$$

Then
* $V_{\mathbb{A}_f} \xrightarrow{\eta} \hat{T}(A) = H_1^{et}(A, \mathbb{A}_f) \simeq H_1(A(\mathbb{C}),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{A}_f \overset{\alpha}{\simeq} V \otimes \mathbb{A}_f$

gives an element of $GSp_{2g}(\mathbb{A}_f)$

* Hodge filtration : $\omega_{A^\vee} \hookrightarrow H_1^{dR}(A/\mathbb{C}) = \Lambda \otimes \mathbb{C} \twoheadrightarrow \mathrm{Lie}_A$

$\Lambda \otimes \mathbb{R} \subseteq \Lambda \otimes \mathbb{C}$, $\Lambda \otimes \mathbb{R} \xrightarrow{\simeq} \mathrm{Lie}_A$: gives $\Lambda \otimes \mathbb{R}$ a complex structure.

So Hodge filtration $\rightsquigarrow$ complex structure on $\Lambda \otimes \mathbb{R}$

$\rightsquigarrow J \in Sp_{2g}(\mathbb{R})$ & $J^2 = -I$ & $\psi(-, J-)$ is positive def.

$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (or neg. def b/c α is up to ± 1)

$\rightsquigarrow$ get $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (i I_g) \in \mathfrak{h}_g^{\pm}$. (exercise)