Étale cohomology of Shimura varieties

§1. Satake isomorphism

Reference: Cartier's article in Corvallis.

division alg $/\mathbb{Q}_p$ $\downarrow$

G unramified group $/\mathbb{Q}_p$
- quasi-split, i.e. $\exists$ a Borel subgp $/\mathbb{Q}_p$ e.g. $D^\times$ is not quasi-split
- split over an unram. ext'n i.e. $G_{\mathbb{Q}_{p^N}}$ has a split max'l torus.

$\rightsquigarrow$ G extends to a group scheme $\mathcal{G}$ over $\mathbb{Z}_p$

$K := \mathcal{G}(\mathbb{Z}_p)$ e.g. $G = \mathrm{Res}_{\mathbb{Q}_{p^N}/\mathbb{Q}_p} GL_n$, $\mathcal{G} = \mathrm{Res}_{\mathbb{Z}_{p^N}/\mathbb{Z}_p} GL_n$.

- $\mathcal{H}_{K_G} := C_c^\infty(K\backslash G(\mathbb{Q}_p)/K, \mathbb{C})$ unramified Hecke algebra.

$\mathrm{vol}(K) = 1$. convolution algebra

$f_1, f_2 \in \mathcal{H}_{K_G} \rightsquigarrow$ $$f_1 * f_2(g) = \int_{G(\mathbb{Q}_p)} f_1(gh^{-1}) f_2(h)\, dh$$

Theorem (Satake) $\mathcal{H}_{K_G}$ is a commutative algebra (will give a description soon)

Cor: If π_p is an irred adm. rep'n of $G(\mathbb{Q}_p)$ and if $\pi_p^{K_p} \neq 0$,

then $\dim \pi_p^K = 1$ & $\mathcal{H}_{K_G}$ acts by a character.

Proof: $\mathcal{H}_{K_G}$ acts on π_p^K by $f \in \mathcal{H}_{K_G}$, $v \in \pi_p^K$

$$\rightsquigarrow f \cdot v := \int_{\mathbb{Q}_p} f(g)\, \pi(g)(v)\, dg$$

π_p irred $\Rightarrow$ π_p^K as a (fin. dim'l) $\mathcal{H}_{K_G}$-module is irred

$\Rightarrow \dim \pi_p^K = 1$ & $\mathcal{H}_{K_G}$ acts by a character

Fact: π_p an irred. rep'n of $G(\mathbb{Q}_p)$, $\pi_p^{K_p} \neq 0 \Leftrightarrow \pi_p$ is an unram. principal series, $\pi_p = \text{n-}\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi$

$$\begin{tikzcd} B(\mathbb{Q}_p) \arrow[r, two heads] & T(\mathbb{Q}_p) \arrow[r] \arrow[rr, bend right, "\chi"'] & T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \arrow[r] & \mathbb{C}^\times \end{tikzcd}$$

(Here $\text{n-}\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi \cdot \delta_B^{\frac{1}{2}}$, δ_B = modulus char of B

remark: our convention follows most literatures e.g. Bump, Goldfeld-Hundley, is $\delta_B^{-\frac{1}{2}}$ of Bushnell-Henniart

E.g. $G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_{p^r}) \supseteq B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \supseteq T \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$$\delta_B^{\frac{1}{2}} \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} = |t_1|^{\frac{n-1}{2}} \cdot |t_2|^{\frac{n-3}{2}} \cdots |t_n|^{\frac{1-n}{2}}$$

Proof of "$\Leftarrow$" $G(\mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot K$.

If $\varphi \in \left(n\text{-}\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi\right)^K \rightsquigarrow \varphi : G(\mathbb{Q}_p) \longrightarrow \mathbb{C}$

s.t. $\varphi(bk) = \chi(b)\delta_B^{\frac{1}{2}}(b)\varphi(1)$ $\swarrow$ χ is trivial here

Note that $b \in B(\mathbb{Q}_p)$ is well-defined up to $B(\mathbb{Z}_p)$

So $$\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi\right)^K \xrightarrow{\simeq} \mathbb{C}$$
$$\varphi \longmapsto \varphi(1)$$

When $\varphi(1) = 1$, this is called the <u>spherical vector</u>.

<u>Example</u>: $G = GL_n(\mathbb{Q}_q)$, $\chi = \chi_1 \times \cdots \times \chi_n : T(\mathbb{Q}_q) \longrightarrow \mathbb{C}^\times$ $\quad q = p^?$

$\alpha_i := \chi_i(p)$ and $\chi_i|_{\mathbb{Z}_q^\times} = \text{triv.}$

For $g_r = \begin{pmatrix} \overbrace{p \cdots p}^{r} & \\ & 1 \cdots 1 \end{pmatrix}$ & $T_r = \mathbb{1}_{Kg_rK}$ & φ the spherical vector

$$T_r(\varphi)(1) = \int_{GL_n(\mathbb{Q}_q)} \mathbb{1}_{Kg_rK}(g) \underbrace{(\pi(g)\varphi)(1)}_{\substack{\parallel \\ \varphi(g)}}$$

Need to compute for each coset Kg_rK/K

$$\overset{\substack{r=2 \\ n=3}}{=} \sum_{a,b \in \mathbb{F}_p} \varphi\begin{pmatrix} p & & a \\ & p & b \\ & & 1 \end{pmatrix} + \sum_{a \in \mathbb{F}_p} \varphi\begin{pmatrix} p & a & \\ & 1 & \\ & & p \end{pmatrix} + \varphi\begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix}$$

$$= q^2\left(q^{-1}\alpha_1 \cdot \alpha_2\right) + q \cdot \left(q^{-1}\alpha_1 \cdot q\alpha_3\right) + \alpha_2 \cdot q\alpha_3$$

$$= q\left(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3\right)$$

in general

$$= q^{\frac{1}{2}r(n-r)} \sum_{a_1 < \cdots < a_r} \alpha_{a_1} \cdots \alpha_{a_r}$$

<u>Summary</u> In this case, $\mathcal{H}k_G$ action on $\left(n\text{-}\mathrm{Ind}_{B_n(\mathbb{Q}_q)}^{GL_n(\mathbb{Q}_q)} \chi\right)^K$ is determined by T_r acts by $q^{\frac{1}{2}r(n-r)} \cdot$ (elementary r^{th} symmetric polynomial in $\chi_1(p), \dots, \chi_n(p)$).

- $T \subseteq B \subseteq G / \mathbb{Q}_p$.

$$W := N(T_{\bar{\mathbb{Q}}_p})/T_{\bar{\mathbb{Q}}_p} = \text{absolute Weyl group} \overset{\mathrm{Gal}_{\mathbb{Q}_p} \curvearrowright}{} \supseteq W_0 = W^{\mathrm{Gal}_{\mathbb{Q}_p}} = \text{relative Weyl group}$$

Example : $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \subseteq B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subseteq G / \mathbb{Q}_{p^m}$

$W = (S_n)^m \circlearrowleft$ Frob rotate factors $\supseteq W_0 =$ diagonal S_n.

Statement of Satake isomorphism:

$$\begin{tikzcd} & B \arrow[dl, "\pi"'] \arrow[dr, hook, "i"] & \\ T & & G \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sat}: C_c^\infty(G(\mathbb{Q}_p)/\!/G(\mathbb{Z}_p), \mathbb{C}) \arrow[r, "i^*"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \arrow[ddrr, "{\substack{\cong \\ \text{algebraic isom.} \\ \text{compatible w/ convolution}}}"'] & C_c^\infty(B(\mathbb{Q}_p)/\!/B(\mathbb{Z}_p), \mathbb{C}) \arrow[r, "\pi_!"] & C_c^\infty(T(\mathbb{Q}_p)/\!/T(\mathbb{Z}_p), \mathbb{C}) \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\ f & & {} \\ & & C_c^\infty(T(\mathbb{Q}_p)/\!/T(\mathbb{Z}_p), \mathbb{C})^{W_0} \end{tikzcd}$$

Explicitly, $\left(\mathrm{Sat}(f)\right)(t) := \delta_B^{\frac{1}{2}}(t) \int_{N(\mathbb{Q}_p)} f(tn)\,dn = \delta_B^{-\frac{1}{2}}(t) \int_{N(\mathbb{Q}_p)} f(nt)\,dn$

Example: $G = \mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} GL_n$

$$C_c^\infty\left(G(\mathbb{Q}_p)/\!/G(\mathbb{Z}_p), \mathbb{C}\right) \overset{\mathrm{Sat}}{\cong} C_c^\infty\left(T(\mathbb{Q}_p)/T(\mathbb{Z}_p), \mathbb{C}\right)^{W_0}$$

$$\cong C_c^\infty\left(\left(\mathbb{Q}_{p^r}^\times/\mathbb{Z}_{p^r}^\times\right)^n, \mathbb{C}\right)^{W_0}$$

$$= \mathbb{C}[X_1^{\pm1}, \dots, X_n^{\pm1}]^{W_0}$$

$$= \mathbb{C}[\sigma_1, \dots, \sigma_{n-1}, \sigma_n^{\pm1}] \quad \text{where } \sigma_i = \sum_{a_1 < \dots < a_i} X_{a_1} \cdots X_{a_i}$$

$$\mathbb{1}_{G(\mathbb{Z}_p)\begin{pmatrix} \overbrace{p \ddots p}^{r} & \\ & 1 \ddots 1 \end{pmatrix} G(\mathbb{Z}_p)} \longmapsto p^{\frac{1}{2}r(n-r)} \sigma_r$$

In terms of earlier computation, the eigenvalue of $\mathrm{Sat}^{-1}(\sigma_r)$ on the spherical vector is the r^{th} symmetric polynomial in $\chi_1(p), \dots, \chi_n(p)$

§2 Dual group and unramified LLC

- $T \subseteq B \subseteq G_{/F} \rightsquigarrow (X^*(T), \Phi, \Delta)$, $(X_*(T), \Phi^\vee, \Delta^\vee)$

E.g. $GL_{n,F} \supseteq \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \supseteq \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}_{/F}$ A_{n-1} ∘—∘— ⋯ —∘—∘

$$\rightsquigarrow \Big(X^*(T)=\mathbb{Z}^n,\ \Phi=\{\alpha_i-\alpha_j\ ;\ i\neq j\},\ \Delta=\{\alpha_1-\alpha_2,\cdots,\alpha_{n-1}-\alpha_n\}$$

$$\alpha_i=(0,\cdots,\overset{i\text{th}}{\overset{\downarrow}{1}},0,\cdots 0)$$

$$X_*(T)=\mathbb{Z}^n,\ \Phi^\vee=\{\alpha_i^\vee-\alpha_j^\vee\ ;\ i\neq j\},\ \Delta=\{\alpha_1^\vee-\alpha_2^\vee,\cdots,\alpha_{n-1}^\vee-\alpha_n^\vee\}\Big)$$

$$G:=\mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}(GL_{n,\mathbb{Q}_{p^r}}),\quad X^*(T)=\underset{\substack{\circlearrowleft\\ \text{Frob. permutation}}}{(\mathbb{Z}^n)^r},\ \Phi=\{\alpha_i^{(s)}-\alpha_j^{(s)}\ ;\ i\neq j,\ s=1,\cdots,r\}$$

$$\Delta=\left\{\alpha_i^{(s)}-\alpha_{i+1}^{(s)}\ ;\ \begin{matrix} i=1,\cdots,n-1\\ s=1,\cdots,r\end{matrix}\right\}$$

$$X_*(T)=(\mathbb{Z}^n)^r,\ \Phi^\vee=\{\alpha_i^{(s),\vee}-\alpha_j^{(s),\vee}\ ;\ i\neq j,\ s=1,\cdots,r\},\ \Delta^\vee=\left\{\alpha_i^{(s),\vee}-\alpha_{i+1}^{(s),\vee}\ ;\ \begin{matrix} i=1,\cdots,n-1\\ s=1,\cdots,r\end{matrix}\right\}$$

$$U_{n,\mathbb{Q}_p}\ \text{unramified}\quad X^*(T)=\underset{\circlearrowleft}{\mathbb{Z}^n}\qquad \Phi=\{\alpha_i-\alpha_j\ ;\ i\neq j\}$$

$$\mathrm{Frob}_p\ \text{acts by}\ (a_1,\cdots,a_n)\mapsto(-a_n,\cdots,-a_1)$$

$$\begin{tikzcd}[column sep=small] \Delta=\{ & \alpha_1-\alpha_2, \arrow[rrr, leftrightarrow, bend right=40, "\mathrm{Frob}_p"'] & \alpha_2-\alpha_3, \arrow[r, leftrightarrow, bend right] & \cdots,\ \alpha_{n-2}-\alpha_{n-1}, & \alpha_{n-1}-\alpha_n & \} \end{tikzcd}$$

$$\rightsquigarrow \text{dual group}\ (\hat{T}\subset\hat{B}\subset\hat{G})$$

$$\text{s.t. } (X^*(T),\Phi,X_*(T),\Phi^\vee,\Delta,\Delta^\vee)\cong(X_*(\hat{T}),\Phi^\vee(\hat{T}),X^*(\hat{T}),\Phi(\hat{T}),\Delta^\vee(\hat{T}),\Delta(\hat{T}))$$

E.g.

- $G=GL_n/_F \rightsquigarrow \hat{G}=GL_n(\mathbb{C})\supseteq\hat{B}=\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\supseteq\hat{T}=\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$
- $G=\mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}GL_n$, $\hat{G}=GL_n(\mathbb{C})\times GL_n(\mathbb{C})\times\cdots\times GL_n(\mathbb{C})$

$$\begin{tikzcd}[column sep=small] GL_n(\mathbb{C}) \arrow[r, bend right] & \times\, GL_n(\mathbb{C}) \arrow[r, bend right] & \times\cdots \arrow[r, bend right] & \times\, GL_n(\mathbb{C}) \arrow[lll, bend left=40] \end{tikzcd}$$

Frob_p cyclic permute factors

- If $G=\mathrm{Res}_{F/\mathbb{Q}}G_0$, $\hat{G}=\mathrm{Ind}_{Gal_{\mathbb{Q}}}^{Gal_F}\hat{G}_0 \circlearrowleft Gal_{\mathbb{Q}}$
$$\mathrel{\rotatebox{90}{$=$}}\ \hat{G}_0\times\cdots\times\hat{G}_0\quad [F:\mathbb{Q}]\ \text{copies.}$$
- $G=U_n/\mathbb{Q}_p$, $\hat{G}=GL_{n,\mathbb{C}} \circlearrowleft \sigma:\ X=\begin{pmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{pmatrix}{}^tX^{-1}\begin{pmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{pmatrix}^{-1}$

$$\sigma\ \text{sends}\ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\ \mapsto\ \begin{pmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{pmatrix}\begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix}\begin{pmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} t_n^{-1} & & \\ & \ddots & \\ & & t_1^{-1} \end{pmatrix}$$

Why the signs? Preserves the "positive direction" in each simple root space

$$\text{e.g.}\quad \begin{pmatrix} & 1\\ 1 & \end{pmatrix}{}^t\begin{pmatrix} 1 & a\\ & 1\end{pmatrix}^{-1}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}=\begin{pmatrix} 1 & -a\\ & 1\end{pmatrix}$$

$$\text{but}\quad \begin{pmatrix} & 1\\ -1 & \end{pmatrix}{}^t\begin{pmatrix} 1 & a\\ & 1\end{pmatrix}^{-1}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}=\begin{pmatrix} 1 & a\\ & 1\end{pmatrix}$$

The Langlands dual $G_{/F}$ is ${}^LG := \hat{G} \rtimes \underbrace{\mathrm{Gal}(\bar{F}/F)}_{}$

often, we replace this by $\mathrm{Gal}(E/F)$, where E = an ext'n of F over which G splits.

Now, let G be unramified $/\mathbb{Q}_p \rightsquigarrow \underset{\substack{\circlearrowleft \\ \sigma = \mathrm{Frob}}}{X^*(T)} \supseteq \Phi \supseteq \Delta$

$\rightsquigarrow \hat{G} \rtimes \langle \sigma \rangle = {}``{}^LG"$ (For simplicity, one may ignore the σ-action on first reading.)

$$\begin{tikzcd}
\mathcal{H}k_G = C_c^\infty\left(G(\mathbb{Q}_p) /\!/ G(\mathbb{Z}_p), \mathbb{C}\right) \arrow[d, "\mathrm{Sat}", "\cong"'] \\
C_c^\infty\left(T(\mathbb{Q}_p) / T(\mathbb{Z}_p), \mathbb{C}\right)^{W_0} \arrow[d, "\cong"] \\
\mathbb{C}\left[X_*(T)^{\sigma=1}\right]^{W_0} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{C}\left[X^*(\hat{T})^{\sigma=1}\right]^{W_0} \arrow[d, phantom, sloped, "\cong", "\text{when } G \text{ is split, i.e. } \sigma \text{ acts trivially on } \hat{T}"] \arrow[dddd, bend right=60, equal, "\text{isom. in general}"'] \\
\mathbb{C}[\hat{T}]^{W_0} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{C}[\hat{T}/W_0] \arrow[d, phantom, sloped, "\cong", "\text{b/c every (semisimple regular) element can be}"] \\
\mathbb{C}[\hat{G}/\mathrm{Ad}\,\hat{G}] \arrow[d, equal, "\text{if } \sigma \text{ is trivial}"] \\
\mathbb{C}[\underbrace{\hat{G}\sigma/\mathrm{Ad}\,\hat{G}}_{}] \quad \sigma : \text{geometric Frobenius}
\end{tikzcd}$$

$$\mathrm{Ad}_h(g\sigma) = hg\sigma h^{-1} = \underbrace{hg\sigma(h)^{-1}}_{\sigma\text{-twisted conj.}} \cdot \sigma$$

$$\begin{tikzcd}
\mathcal{H}k_{GL_n} = C_c^\infty\left(GL_n(\mathbb{Q}_p) /\!/ GL_n(\mathbb{Z}_p), \mathbb{C}\right) \arrow[d, "\mathrm{Sat}", "\cong"'] \\
C_c^\infty\left(\left(\mathbb{Q}_p^\times / \mathbb{Z}_p^\times\right)^n, \mathbb{C}\right)^{S_n} \arrow[d, "\cong"] \\
\mathbb{C}[\mathbb{Z}^n]^{S_n} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{C}[\mathbb{Z}^n]^{S_n} \arrow[d, phantom, sloped, "\cong", "\widehat{GL_n} = GL_{n/\mathbb{C}} \supseteq \mathbb{G}_m^n"] \\
\mathbb{C}[\mathbb{G}_m^n]^{S_n} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{C}[\mathbb{G}_m^n / S_n] \arrow[d, phantom, sloped, "\cong", "\leftarrow \text{eigenvalues of a complex matrix}"] \\
\mathbb{C}[GL_n / \mathrm{Ad}(GL_n)]
\end{tikzcd}$$

<u>Summary</u>: Unramified principal series $\pi_p = n\text{-}\mathrm{Ind}_B^G \chi$

$\rightsquigarrow$ $\mathcal{H}k_G$-action on $\pi_p^{G(\mathbb{Z}_p)}$ by a homomorphism $\mathcal{H}k_G \xrightarrow{\varphi_{\pi_p}} \mathbb{C}$

$$\overset{\text{Satake isom.}}{\rightsquigarrow} \varphi_{\pi_p} : \mathbb{C}\left[\hat{G}\sigma / \mathrm{Ad}\,\hat{G}\right] \longrightarrow \mathbb{C}$$

$\rightsquigarrow$ a point $\gamma_{\pi_p} \in \hat{G}\sigma \subseteq \hat{G} \rtimes \langle\sigma\rangle = {}^L G$

up to $\hat{G}$-conjugacy.

In the example, $\varphi_{\pi_p} : \mathcal{H}k_G \longrightarrow \mathbb{C}$

$$\mathbb{1}_{GL_n(\mathbb{Z}_p)\,\mathrm{diag}\{\overbrace{p,\dots,p}^{r},1,\dots,1\}\,GL_n(\mathbb{Z}_p)} \longmapsto p^{\frac{r(n-r)}{2}} \cdot r^{th}\text{ symmetric power in } \chi_1(p),\dots,\chi_n(p)$$

$\cong$

$p^{\frac{r(n-r)}{2}} \cdot \sigma_r$ in $t_1,\dots,t_n$ for $\hat{T} = \mathbb{G}_m^n$

$$\Rightarrow \gamma_{\pi_p} \sim \begin{pmatrix} \chi_1(p) & & \\ & \ddots & \\ & & \chi_n(p) \end{pmatrix} \text{ up to } GL_n(\mathbb{C})\text{-conjugacy.}$$

This defines a passage : unramified π_p $\rightsquigarrow$ unramified rep'n $Gal_{\mathbb{Q}_p} \to Gal_{\mathbb{F}_p} \to {}^L G$

$$\begin{tikzcd} Gal_{\mathbb{F}_p} \arrow[r] & {}^L G \\ \sigma \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \gamma_{\pi_p}\sigma \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

Compatibility with Weil restriction:

G_0 split group over $\mathbb{Q}_{p^r}$ & $G := \mathrm{Res}_{\mathbb{Q}_{p^r}/\mathbb{Q}_p} G_0$

$\hat{G}_0 =$ Langlands dual gp , $\hat{G} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^r}}}^{G_{\mathbb{Q}_p}} \hat{G}_0 \rtimes \langle\sigma_p\rangle$

$$= \left(\hat{G}_0 \times \cdots \times \hat{G}_0\right) \rtimes \langle\sigma_p\rangle$$

$$\begin{tikzcd} \hat{G}_0 \arrow[r, "\sigma_p"] & \hat{G}_0 \arrow[r, "\sigma_p"] & \cdots \arrow[r, "\sigma_p"] & \hat{G}_0 \arrow[lll, bend left, "\sigma_p"] \end{tikzcd}$$

π_v an unramified rep'n of $G_0(\mathbb{Q}_{p^r}) = G(\mathbb{Q}_p)$

$\hookrightarrow$ $\gamma_{\pi_v}\sigma_{p^r} \in \hat{G}_0 \rtimes \langle\sigma_{p^r}\rangle$ up to conj by $\hat{G}_0$

$\hookrightarrow$ $(\gamma_{\pi_v}^{(1)}, \dots, \gamma_{\pi_v}^{(r)})\,\sigma_p \in \hat{G} \rtimes \langle\sigma_p\rangle$ $\nearrow$ up to conj by $\hat{G}$

$$\mathrm{Ad}_{(g_1,\dots,g_r)}(h_1,\dots h_r)\sigma_p = (g_1 h_1 g_{r-1}^{-1}, g_2 h_2 g_1^{-1}, \dots\dots g_r h_r g_{r-1}^{-1})$$

So $h_r h_{r-1} \cdots h_1$ up to $\hat{G}_0$-action is an inv.

These are compatible in the sense that

$$\left(\left(\gamma^{(1)}, \dots, \gamma^{(r)}\right)\sigma\right)^r = \left(\gamma^{(1)}\gamma^{(r)}\gamma^{(r-1)} \cdots \gamma^{(2)}, \gamma^{(2)}\gamma^{(1)}\gamma^{(r)} \cdots \gamma^{(3)}, \dots\right)\sigma^r$$

$$\left(\left(\gamma_{\pi_v}, \ldots, \gamma_{\pi_v}\right)\sigma_p\right) = \left(\underbrace{\gamma_{\pi_v}\gamma_{\pi_v}\gamma_{\pi_v}\cdots\gamma_{\pi_v}}_{\substack{\uparrow \\ \text{each one of them is } \hat{G}_0\text{-conj to } \gamma_{\pi_v}.}}, \gamma_{\pi_v}\gamma_{\pi_v}\gamma_{\pi_v}\cdots\gamma_{\pi_v}, \ldots\right)\sigma_p$$

In practice, up to $\hat{G}_0$-conjugacy, can take

$$(\gamma_{\pi_v}^{(1)}, \ldots, \gamma_{\pi_v}^{(r)})\sigma_p = (\gamma_{\pi_v}, 1, \ldots, 1)\sigma_p.$$

Galois rep'ns associated to modular forms

Fix an isom. $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$

f wt k, level N $\longleftrightarrow$ π_f autom. rep'n of $GL_2(\mathbb{A})$. central char $|\cdot|^{2-k}$

for $p \nmid N$, $T_{p,1}$-eigenvalue on π_f is $a_p(f)$

$T_{p,2}$-eigenvalue on π_f is $p^{k-2}\psi(p)$ $\longleftrightarrow$ when wt 2, $T_{p,2}$ acts by root of unity

$\Big\downarrow$

Associated Galois rep'n $\rho_f : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\bar{\mathbb{Q}}_\ell)$

s.t. for $p \nmid N\ell$, $\mathrm{charpoly}(\rho_f(\mathrm{Frob}_p)) = x^2 - a_p(f)x + p^{k-1}\psi(p)$

* One sees a twist here :

$$T_{p,1} = a_p, \; T_{p,2} = p^{k-2}\psi(p) \implies \mathrm{charpoly}(\gamma_{\pi_p}) = x^2 - p^{-\frac{1}{2}}a_p(f)x + p^{k-2}\psi(p)$$

$$\rho_f(\mathrm{Frob}_p) := \gamma_{\pi_p} \cdot \begin{pmatrix} p^{\frac{1}{2}} & \\ & p^{\frac{1}{2}} \end{pmatrix}$$

Kottwitz: Given a Shimura datum $(G, X) \rightsquigarrow \mu : \mathbb{G}_m \to G_{\mathbb{C}}$ def'd over $\overset{\text{reflex field}}{\overset{\downarrow}{E}}$.

Can conjugate so that μ is a dominant cocharacter in $X_*(T) = X^*(\hat{T})$

$\rightsquigarrow$ highest weight rep'n $r_\mu : \hat{G} \to GL(V_\mu)$

Fact r_μ extends to a rep'n $r_\mu : \hat{G} \rtimes \mathrm{Gal}_{\bar{\mathbb{Q}}/E} \to GL(V_\mu)$

Given an unramified rep'n $\pi_p \rightsquigarrow \gamma_{\pi_p}\sigma \in \hat{G} \rtimes \langle\sigma\rangle$

$\rightsquigarrow$ For a p-adic place v of E, get an unram. rep'n.

$$\begin{tikzcd}
\mathrm{Gal}_{E_v} \arrow[r] & \hat{G} \rtimes \mathrm{Gal}_{\bar{\mathbb{Q}}_p/E_v} \arrow[r] & GL(V_\mu)(\bar{\mathbb{Q}}_\ell) \\
\sigma_v = \sigma_p^m \arrow[r, mapsto] & (\gamma_{\pi_p}\sigma)^m \arrow[r, mapsto] & r_\mu\left((\gamma_{\pi_p}\sigma)^m\right)
\end{tikzcd}$$

<u>Twist</u>:

$$\begin{aligned} \chi : \mathrm{Gal}_{E_v} &\longrightarrow \bar{\mathbb{Q}}_\ell^\times \\ \sigma_v &\longmapsto (\sqrt{p})^{\dim Sh_G \cdot [E_v:\mathbb{Q}_p]} \end{aligned}$$

<u>Expectation</u>: Given a "$\underset{\sim}{\text{nice}}$" automorphic rep'n π,

$$H^{\mathrm{mid}}_{\text{ét}}\left(Sh_G(K_f), \underbrace{\underline{V}(\lambda)}_{\text{some } \ell\text{-adic local system}}\right) = \bigoplus_{\substack{\pi \\ H^{\mathrm{mid}}(\mathfrak{g}, K_\infty; \pi_\infty \otimes V(\lambda)) \neq 0}} \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes W(\pi) \curvearrowleft \mathrm{Gal}_E$$

$W(\pi) \simeq V_\mu$ is unramified at where K_p is "hyperspecial" $\left(= G(\mathbb{Z}_p)\right)$ &

$$\begin{aligned} \mathrm{Gal}_{E_v} &\longrightarrow GL(V_\mu) \\ \sigma_v &\longmapsto r_\mu\left((\gamma_{\pi_p}\sigma)^m\right) \otimes \chi(\sigma_v). \end{aligned}$$

<u>Example</u>: $F/\mathbb{Q}$ totally real

$G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$ Hilbert modular variety , reflex field $= \mathbb{Q}$

π autom. rep'n of parallel wt $(2,2,\cdots;2;2)$

$$\mu = ((1,0),(1,0),\cdots,(1,0)) \in X^*(\hat{T}) = X^*((\mathbb{G}_m \times \mathbb{G}_m)^d)$$

$$\rightsquigarrow V_\mu = \mathrm{std} \otimes \cdots \otimes \mathrm{std} : \hat{G} = GL_2^d \longrightarrow GL_{2^d}$$

= a tensorial induction of a rep'n of GL_2 from Gal_F to $\mathrm{Gal}_{\mathbb{Q}}$

Classical way to describle this:

$\exists$ a Galois rep'n $\rho_\pi : \mathrm{Gal}_F \longrightarrow GL_2(\bar{\mathbb{Q}}_\ell)$ (viewing π as an autom rep'n of $GL_{2/F}(\mathbb{A}_F)$)

s.t. for v a "good" place

$$\mathrm{charpoly}(\rho_\pi(\mathrm{Frob}_v)) = x^2 - a_v x + N(v)\psi(v) = 0$$

Then

$$H^{[F:\mathbb{Q}]}\left(Sh_G(K_f), \bar{\mathbb{Q}}_\ell\right)_{\mathrm{cusp}} \approx \bigoplus_{\substack{\pi \text{ cusp autom,} \\ \mathrm{wt}(2,\cdots 2;2)}} \pi_f^{K_f} \otimes \Big(\underbrace{\otimes\text{-}\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \rho_\pi}_{\text{tensorial induction.}}\Big)$$

$$\left(\otimes\text{-}\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \rho_\pi\right)\Big|_{\mathrm{Gal}_{F^{\mathrm{Gal}}}} = \rho_\pi$$