Shimura varieties of PEL type

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1 Shimura datum related to real semisimple algebra

Suppose we have a finite-dimensional semisimple $\mathbb{R}-$algebra B.

Definition 1.1. An involution $*$ of B is an automorphism(as a $\mathbb{R}-$vector space) of B such that $*^2 = id$ and $(ab)^* = b^*a^*$.

Remark 1.2. All semisimple algebras equipped with involutions over algebraically closed field are basically built from 2 types: matrix algebra equipped with the composition of transpose and conjugation, product of two matrix algebra equipped with interchanging two factors.

In the following context, modules are automatically finitely generated $B-$modules.

Theorem 1.3. *The following are equivalent conditions on the involution* $*$.

(1) Every $B-$*module carries some positive Hermitian form;*

(2) For every faithful $B-$*module* V *(i.e.* $B \to End(V)$ *is injective) we have* $tr(xx^*) > 0$ *for all nonzero* $x \in B$*;*

(3) $tr_{B/\mathbb{R}}(xx^*) > 0$ *for all* $x \in B$*;*

(4) There exists a $B-$*module* V *such that* $tr(xx^*; V) > 0$ *for all nonzero* $x \in B$*;*

(5) There exists a faithful positive Hermitian $B-$*module.*

The Hermitian form here on $B-$*module means a symmetric real-valued bilinear form* (v, w) *on a* $B-$*module such that* $(bv, w) = (v, b^*w), \forall b \in B, \forall v, w \in V$*. A* $B-$*module equipped with a Hermitian form is called a Hermitian* $B-$*module.*

Involutions satisfying the above conditions are called positive involutions.

Remark 1.4. From (5), we can see tensor product of positive involution is still a positive involution.

Proof. See **Lemma 2.2.** [Kot92] □

Proposition 1.5. *Suppose* B *is equipped with a positive involution* $*$*. Then two positive definite Hermitian* $B-$*modules are isomorphic as Hermitian* $B-$*modules if and only if they are isomorphic as* $B-$*modules*

Proof. It comes from the following facts:

(1) any positive definite Hermitian $B-$module can be written as a direct sum of definite Hermitian $B-$modules which are irreducible as $B-$modules;

(2) the real vector space of Hermitian forms on an irreducible $B-$module is one-dimensional.

For details, you may check Section 2 of [Kot92]. $\square$

Given a semisimple finite-dimensional $\mathbb{R}-$algebra C with involution $*$, suppose there exists an $\mathbb{R}-$algebra homomorphism $h : \mathbb{C} \to C$ such that $h(z)^* = h(\bar{z})$(such homomorphism will be called $*-$homomorphism). Thus $h(i)* = h(-i) = -h(i)$. So we can define another involution of C: $\iota(x) := h(i)^{-1}x^*h(i)$. Suppose ι is a positive involution.

Define G to be the algebraic group over $\mathbb{R}$ whose points in an $\mathbb{R}-$algebra R are given by $G(R) := \{x \in C \otimes_{\mathbb{R}} R | xx^* \in R^\times\}$. Note that for $z \in \mathbb{C}^\times$. $h(z)h(z)^* = h(z)h(\bar{z}) = h(z\bar{z}) = z\bar{z} \in \mathbb{R}^\times$. Thus $h(\mathbb{C}^\times) \subset G(\mathbb{R})$.

Theorem 1.6. (G, h) *satisfies the first two axioms of Shimura datum*

(1) $Ad \circ h$ *gives a Hodge structure* $Lie(G)_{\mathbb{R}}$ *of type* $(1, -1), (0, 0), (-1, 1)$*;*

(2) $Ad(h(i))$ *is a Cartan involution of* G^{ad}*.*

Proof. Axiom (1) just comes from the fact that h is a restriction of $\mathbb{R}-$algebra homomorphism $\mathbb{C} \to C$.

Firstly, $h(i)^2 = h(i^2) = h(-1)$ lies in the center of C. Thus $h(i)^2$ lies in the center of $G(\mathbb{R})$. Hence $Ad(h(i))^2 = Ad(h(i)^2) = id$.

Note that $G^{ad}(\mathbb{R}) = \{x \in C | xx^* = 1\}$. Now it suffices to show

$$\{x \in C \otimes_{\mathbb{R}} \mathbb{C} | xx^* = 1, h(i)^{-1}\bar{x}h(i) = x\}$$

is compact. Note that this group is a closed subgroup of

$$\{x \in C \otimes_{\mathbb{R}} \mathbb{C} | x\iota(\bar{x}) = 1\}.$$

By **Remark** 1.4, we know that $\iota(\bar{\cdot})$ is a positive resolution of $C \otimes_{\mathbb{R}} \mathbb{C}$. By choosing a faithful positive Hermitian $C \otimes_{\mathbb{R}} \mathbb{C}-$module, we are able to realize this group as a closed subgroup of some orthogonal group. Thus we are done. $\square$

In our problems, the above triple $(C, *, h)$ usually comes from the following way.

Let B be a semisimple finite dimensional $\mathbb{R}-$algebra with positive involution $*$. Let V be a left $B-$module equipped with a non-degenerate real-valued alternating bilinear form $(\cdot, \cdot)$, s.t. $(bv, w) = (v, b^*w)$(such module will be called skew-Hermitian module too). Set $C := End_B(V)$. For $c \in C$, c^* is defined to satisfy $(cv, w) = (v, c^*w)$. Note that c^* still lies in C:

$$(v.c^*bw) = (cv, bw) = (b^*cv, w) = (cb^*v, w) = (b^*v, c^*w) = (v, bc^*w).$$

Let $h : \mathbb{C} \to End_B(V)$ be an $\mathbb{R}-$algebra homomorphism such that $(h(z)v, w) = (v, h(\bar{z})w)$. Thus h is a $*-$homomorphism. Let's further assume $(v, h(i)w)$ is positive definite. Then we will get a triple $(C, *, h)$ satisfying the foregoing conditions. In fact, any triple can be constructed in this way.

Proposition 1.7. *Given $(B, *, V, (\cdot,\cdot), h)$ as above, we assume that $(\cdot,\cdot)^{'}$ and $h^{'}$ satisfies the same conditions as $(\cdot,\cdot)$ and h. Assume further the two $B \otimes_{\mathbb{R}} \mathbb{C}-$module structures obtained from h and h' are isomorphic. Then $(V, (\cdot,\cdot), h)$ and $(V, (\cdot,\cdot)^{'}, h^{'})$ are isomorphic as skew-Hermitian $B-$modules. Moreover, the $G(\mathbb{R})-$conjugacy class X_∞ of h is equal to the set of $*-$homomorphisms $h' : \mathbb{C} \to C$ satisfying the following two conditions:*

(1) the form $(v, h'(i)w)$ is positive definite or negative definite.

(2) the two $B \otimes_{\mathbb{R}} \mathbb{C}-$modules structures on V obtained from h and h' are isomorphic.

Proof. Firstly, we can view V as a $B \otimes_{\mathbb{R}} \mathbb{C}-$module by h. In fact, the positive definite form $(v, h(i)w)$ is a $B \otimes_{\mathbb{R}} \mathbb{C}-$Hermitian form where the positive involution of $B \otimes_{\mathbb{R}} \mathbb{C}$ is the tensor product of $*$ and conjugation. We have the analogous assertion for $(\cdot,\cdot)^{'}$ and $h^{'}$ as well.

By **Proposition** 1.5, the two $B \otimes_{\mathbb{R}} \mathbb{C}-$Hermitian forms on V are isomorphic. Thus there exists $c \in C^\times$ such that $h' = Int(c) \circ h$ and $(v, h(i)w) = (cv, h'(i)cw)$. Hence $(v, w) = (cv, cw)'$. So c is an isomorphism between skew-Hermitian $B-$module, i.e. the first statement is proved.

If we multiply $(\cdot,\cdot)$ some nonzero real number, we can use the first part to see there exists $c \in C^\times$ preserving $(\cdot,\cdot)$ up to a scalar such that $Int(c) \circ h = h'$. Thus h' can be conjugated to h by elements of $G(\mathbb{R})$. □

2 Moduli spaces of PEL type

In this section, we will turn to number fields.

Assume F to be a CM field. Let B be a simple algebra with center F(i.e. matrix algebra of central division algebra over F).

Assume $*$ is an involution of B which becomes a positive involution of $B \otimes_{\mathbb{Q}} \mathbb{R}$ after base change. Take a $B-$module V equipped with a non-degenerate alternating form on V such that $\langle bv, w\rangle = \langle v, b^* w\rangle$.

Example 2.1. Suppose $B = F$ and $*$ is the complex conjugation. Take V is an n dimensional $F-$vector space. Fix $\delta \in F^{*=-1}$, a totally imaginary element and $n = r + s$. We can define the non-degenerate alternating form on V as follows:

$$\langle (a_1, ..., a_n), (b_1, ..., b_n)\rangle = Tr_{E/\mathbb{Q}}(\delta(a_1\bar{b}_1 + ... + a_r\bar{b}_r - a_{r+1}\bar{b}_{r+1} - ... - a_n\bar{b}_n))$$

Set $C := End_B(V)$. Then $*$ and $\langle\cdot,\cdot\rangle$ gives an involution $*$ on C.

Define G to be a $\mathbb{Q}-$group whose $R-$points for any $\mathbb{Q}-$algebra R are given by :

$$G(R) = \{g \in C \otimes_{\mathbb{Q}} R | g^* g \in R^\times\} = \{(g, c) \in (C \otimes_{\mathbb{Q}} R)^\times \times R^\times \,|\, \langle gv, gw\rangle = c\,\langle v, w\rangle\}.$$

Example 2.2. Let's still follow the foregoing example. Then $C = Mat_{n\times n}(F)$

Note that $\langle v, g^* gw\rangle = \langle gv, gw\rangle$. Thus G is just $GU(r, s)$ which preserves the alternating form up to a scalar.

Suppose we have a $*-$homomorphism of $\mathbb{R}-$algebra $h : \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R}$ such that $\langle v, h(i)w \rangle$ is symmetric positive definite.

Lemma 2.3. *There exists a decomposition of $B_{\mathbb{C}}-$module $V_{\mathbb{C}} = V_1 \oplus V_2$ such that $h(z)$ coincides multiplication by z on V_1 and multiplication by $\bar{z}$ on V_2.*

Proof. Since h is an $\mathbb{R}-$algebra homomorphism, $h(a+bi) = h(a) + h(bi) = a + bh(i)$. Thus it suffices to determine $h(i)$.

Note that $h(i)^2 = h(i^2) = h(-1) = -1$. So $h(i)$ must be similar to an element of $GL(V_{\mathbb{C}})$ of the form $diag(iI_r, -iI_s)$. Thus $h(z), z \in \mathbb{C}$ are simultaneously similar to $diag(zI_r, \bar{z}I_s)$. Since the image of h centralizes $B_{\mathbb{C}}$, the decomposition is a decomposition of $B_{\mathbb{C}}-$module. □

Assume that $E \subset \mathbb{C}$ is the reflex field of V_1, i.e. the smaller field where V_1 can be defined is E.

Before introducing the moduli spaces, we are going to add more our assumptions on F and B:

· Fix a prime number p at which F and E is unramified and a compact open group $K^p \subset G(\mathbb{A}_f^p)$(labelling p overhead means there's no $p-$factor);

· Assume that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p = B \otimes_F F \otimes_{\mathbb{Q}} \mathbb{Q}_p = B \otimes_F \bigoplus_{\mathfrak{P}_i | p} F_{\mathfrak{P}_i}$ is the direct product of matrix algebra $M_m(\mathbb{Q}_{p^{f_i}})$ equipped with a positive involution;

· Assume there exists an $\mathbb{Z}_{(p)}-$order(labelling with (p) means localization) O_B in B stable under $*$ such that $\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is the direct product of matrix algebra $M_m(\mathbb{Z}_{p_i^f})$;

· Assume there exists an $\mathcal{O}_B-$lattice Λ of V such that he non-degenerate Hermitian form of V takes value in $\mathbb{Z}_{(p)}$ when restricting to Λ and $\Lambda = \Lambda^\vee$.

Before defining the moduli problem, let's introduce some terms of abelian schemes.

For a scheme S, define AV_S^p to be the category of abelian schemes with morphisms as prime-to-p isogenies:

·**Objects** : abelian schemes over S

·**Morphisms** : $Hom_{AV_S^p}(X,Y) := Hom_S(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

A prime-to-p polarization on X is a morphism $\phi : X \to X^\vee$ in AV_S^p such that there exists $n \in \mathbb{N}$ and $p \nmid n$ making $n\phi$ a genuine polarization with degree not divided by p.

For $A \in AV_S^p$, let's define the prime-to-p Tate module

$$\hat{\mathcal{V}}^p(A) := (\varprojlim_{p \nmid n} A[N])) \otimes_{\hat{\mathbb{Z}}^p} \mathbb{A}_f^p.$$

Note that morphisms in AV_S^p will naturally induces morphisms between prime-to-p Tate modules.

Now we are ready to state the moduli problem.

Theorem 2.4. *When K^p is small enough, the following functor:*

$$\mathscr{M}_{K^p}: \qquad Sch/_{\mathcal{O}_{E,(p)}} \longrightarrow Sets$$

$$S \longrightarrow \left\{ \begin{array}{l} (A,\lambda,\eta): \text{ up to } \mathcal{O}_B\text{-prime-to-p isogeny(explained later):} \\ \cdot A, \text{ an object in } AV_S^p \text{ equipped with a faithful } \mathcal{O}_B\text{-action(i.e. an embedding } i:\mathcal{O}_B \hookrightarrow End_{AV_S^p}(A)) \\ \text{satisfyingr Kottwitz condition;} \\ \cdot\lambda : A \rightarrow A^\vee \text{ a prime-to-p polarization s.t the Rosati involution induces } * \text{ on } \mathcal{O}_B. \\ \cdot \text{ On each connected component of } S, \text{ fixing a geometric point } \overline{s}, \eta \text{ is a } \pi_1(S,\overline{s})- \text{ stable } K-\text{orbit of } \mathcal{O}_B- \text{ linear isomorphism:} \\ \qquad \eta_{\bar{s}} : V_{\mathbb{A}_f^p} := V \otimes_{\mathbb{Z}_{(p)}} \mathbb{A}_f^p \cong \hat{\mathcal{V}}^p(A) \\ \text{compatible with Weil pairing and the pairing on } V. \end{array} \right\}$$

is represented by a smooth scheme.

Up to $\mathcal{O}_B$-prime-to-p isogeny : $(A,\lambda,\eta),(A',\lambda',\eta')$ are isomorphic if there exists a prime-to-p isogeny from A to A' commuting with $\mathcal{O}_B - action$ carrying η to η' and λ to a scalar($\in \mathbb{Z}_{(p)}^\times$) multiple of λ'

Kottwitz condition : Let $\alpha_1,...,\alpha_r$ be a $\mathbb{Z}_{(p)}-$basis of $\mathcal{O}_B$. Note that $\alpha_1,...,\alpha_r$ act on $E-$vector space V_1. It's a fact that coefficients of the polynomial $det(X_1\alpha_1+...+X_r\alpha_r;V_1)\in E[x_1,...,x_r]$ lie in $\mathcal{O}_{E,(p)}$. Similarly $\alpha_1,...,\alpha_r$ act on the locally free $\mathcal{O}_S-$module Lie(A). Thus coefficients of $det(X_1\alpha_1+...+X_r\alpha_r;Lie(A))$ lie in $\mathcal{O}_S$. Note that $\mathcal{O}_S$ is an $\mathcal{O}_{E,(p)}-$algebra. The Kottwitz condition is just to require $det(X_1\alpha_1+...+X_r\alpha_r;V_1) = det(X_1\alpha_1+...+X_r\alpha_r;Lie(A))$.

Remark 2.5. Here is the significance of Kottwitz condition. Let E be a finite-dimensional semisimple algebra over a field k and $\alpha_1,...,\alpha_t$ be a $k-$basis for E. For any finite-dimensional $E-$module V, let's define a polynomial $det_V := det(X_1\alpha_1+...+X_r\alpha_r;V)$. Then for any two $E-$module V and W, V is isomorphic to W if and only if $det_V = det_W$.

Kottwitz condition in explicit terms :

Note that $O_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = \prod_i M_m(\mathbb{Z}_{p^{f_i}})$ acts on $\Lambda \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ which induces a decomposition $\Lambda \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \prod_i (\mathbb{Z}_{p^{f_i}}^{\oplus m})^n$.

Recall that we have a decomposition $V_{\mathbb{C}} = V_1 \oplus V_2$ where the decomposition is defined over $\mathcal{O}_E$. Let's pick up a factor $\mathbb{Z}_{p^N}$ of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then we can see the decomposition after base change to $\mathbb{Z}_{p^N}$.

Note that we have the decomposition as follows:

$$\prod_i (\mathbb{Z}_{p^{f_i}}^{\oplus m})^n \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^N} = \prod_i \prod_{\tau:\mathbb{Q}_{p^{f_i}} \to \mathbb{Q}_{p^N}} (\mathbb{Z}_{p^N}^{\oplus m})^n$$
$$= \prod_i \prod_{\tau:\mathbb{Q}_{p^{f_i}} \to \mathbb{Q}_{p^N}} ((\mathbb{Z}_{p^N}^{\oplus m})^{r_{i,\tau}} \oplus (\mathbb{Z}_{p^N}^{\oplus m})^{s_{i,\tau}})$$

where the first summand generates V_1.

Note that over $A_N := S \times_{O_{E,(p)}} \mathbb{Z}_{p^N}$, we have an $O_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^N} = \prod_i \prod_\tau M_m(\mathbb{Z}_{p^N})-$action on $\omega_{A^\vee/S_N} \subset H_1^{dR}(A/S_N)$. Take $e_{i,\tau}$ to be the matrix with $(1,1)-$entry 1 and other entries 0 of index (i,τ). Then we have the following decomposition:

$$\bigoplus_i \bigoplus_\tau (e_{i,\tau}\omega_{A^\vee/S_N})^m = \omega_{A^\vee/S_N} \subset H_1^{dR}(A/S_N) = \bigoplus_i \bigoplus_\tau (e_{i,\tau} H_1^{dR}(A/S_N))^m$$

. The Kottwitz conditions require $rank(e_{i,\tau}\omega_{A^\vee/S_N}) = s_{i,\tau}$.

3 Complex points of the moduli space

Let $X = G(\mathbb{R})-$conjugacy class of homomorphism $h : \mathbb{C}^\times \to G(\mathbb{R})$. Our dream is to show $\mathscr{M}_{K^p}(\mathbb{C}) = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K^p G(\mathbb{Z}_p)$. However, this is not quite correct. The complex points is finite union of Hermitian symmetric space. In the following context, we are going to assume $p \nmid 2$ if $C_\mathbb{R}$ is a product of several copies of $M_n(\mathbb{H})$.

Given a triple (A,λ,η), let's denote $H := H_1(A,\mathbb{Q})$. λ will give H an Note that

$$H \otimes_\mathbb{Q} \mathbb{A}_f^p = H_1(A, \mathbb{A}_f^p) = H_1^{et}(A, \mathbb{A}_f^p) = \hat{\mathcal{V}}^p(A).$$

We claim that for every place v of $\mathbb{Q}$ the skew-Hermitian $B_{\mathbb{Q}_p}-$modules $H_{\mathbb{Q}_v}$ and $V_{\mathbb{Q}_v}$ are isomorphic(the isomorphisms here respect the alternating form).

For those finite places q other than p, it just comes from the restriction of η on $q-$factor. Moreover, by isomorphism as finite places other than p, we see that H and V are isomorphic as $B-$modules.

For v at ∞, we already have $H_\mathbb{R}$ and $V_\mathbb{R}$ are isomorphic as $B_\mathbb{R}-$modules. Recall that we have decomposed $V_\mathbb{C}$ as $V_1 \oplus V_2$. We can do the same thing to $H_\mathbb{R} = Lie(A)$ which admits a complex structure from complex multiplication on the Lie group to obtain the decomposition $H_\mathbb{C} = H_1 \oplus H_2$. Kottwitz condition gives us $H_1 \cong V_1$ as $B_\mathbb{C}-$module. Moreover $V_\mathbb{C}$ and $H_\mathbb{C}$ are isomorphic $B_\mathbb{C}-$modules. Hence we have $H_\mathbb{C}$ and $V_\mathbb{C}$ are isomorphic $B_\mathbb{C} \otimes_\mathbb{R} \mathbb{C}-$modules. Thus we have $H_\mathbb{R}$ and $V_\mathbb{R}$ are isomorphic $B_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}-$modules. Then the isomorphism of skew-Hermitian modules follows from **Proposition** 1.7.

Note that $H^1(\mathbb{Q},G)$(resp. $H^1(\mathbb{Q}_v,G)$) classified isomorphism classes of skew-Hermitian $B-$modules(resp. $B_{\mathbb{Q}_v}-$modules) with same dimension as V. By previous claim, we learn that there is a bijection between isomorphic classes of H_1 and $ker(H^1(\mathbb{Q},G) \to \prod_v H^1(\mathbb{Q}_v,G))$ which consists of finite elements.

Example 3.1. If we take B to be F and V to be a Hermitian vector space over F, the group G will become $GU(V)$. When $dim_F V$ is even, G will satisfies

Hasse principle, i,e. $ker(H^1(\mathbb{Q},G) \to \prod_v H^1(\mathbb{Q}_v,G))$ is trivial. When $dim_F V$ is odd, it's nontrivial.

Pick up representatives $V^{(1)},...,V^{(k)}$ of these isomorphism classes.Then we have:

$$\mathscr{M}_{K^p}(\mathbb{C}) = \bigsqcup_{1\leq i\leq k} \mathscr{M}_{K^p}(\mathbb{C})^{(i)}$$

where $\mathscr{M}_{K^p}(\mathbb{C})^{(i)}$ consists of the triples (A,λ,η) such that $H_1(A,\mathbb{Q}) \cong V^{(i)}$ as a skew-Hermitian $B-$module. Denote $G^{(i)}$ to be the group preserving the skew-Hermitian forms on $V^{(i)}$ up to a scalar.

Now let's investigate $\mathscr{M}_{K^p}(\mathbb{C})^{(i)}$. Now let's choose an isomorphism $\alpha : V^{(i)} \cong H$. Then we have an automorphism of skew-Hermitian module by η and α:

$$V^{(i)}_{\mathbb{A}^p_f} \cong H \otimes_{\mathbb{Q}} \mathbb{A}^p_f \cong \hat{\mathcal{V}}^p(A) \cong V^{(i)}_{\mathbb{A}^p_f}.$$

Note that η is a representative of K^p-orbit. Thus the above identification gives us an element of $G(\mathbb{A}^p_f)/K^p$. Moreover, the complex structure of $H_{\mathbb{R}} = Lie(A)$ gives a complex structure of $V_{\mathbb{R}}$, i.e. an $\mathbb{R}-$algebra homomorphism $h' : \mathbb{C} \to C_{\mathbb{R}}$. By **Proposition** 1.7, h' belongs to X_∞, the $G(\mathbb{R})-$conjugacy classes of h.

Since A is up to prime-to-p isogenies, $H_1(A,\mathbb{Z}_p)$ is well-defined self-dual $\mathcal{O}_B-$lattice in $H_{\mathbb{Q}_p}$.By α, $H_1(A,\mathbb{Z}_p)$ is self-dual up to a scalar. Fix a self-dual $\mathcal{O}_B-$lattice Λ_0 in $V^{(i)}_{\mathbb{Q}_p}$ with stabilizer K_p(existence is assured in section 7 of [Kot92]). Section 7 of [Kot92] tells us there exists $g \in G^{(i)}(\mathbb{R})$ such that $H_1(A,\mathbb{Z}_p) = g\Lambda_0$. In fact g is up to right K_p action.

Define $K := K_pK^p$. Above all, an isomorphism α between H and $V^{(i)}$ will give us a well-defined element in $G^{(i)}(\mathbb{A}_f)/K \times X_\infty$. Note that different choice of α will defer by an automorphism of $V^{(i)}$(with respect the skew-Hermitian form), i.e. an element of $G^{(i)}(\mathbb{Q})$. Hence there is a bijection :

$$\mathscr{M}_{K^p}(\mathbb{C})^{(i)} \leftrightarrow G^{(i)}(\mathbb{Q}) \setminus G^{(i)}(\mathbb{A}_f)/K \times X_\infty.$$

References

[Kot92] Robert E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.