# Shimura varieties of PEL type d'apres Kottwitz

Goal : Define integral model of Shimura varieties of PEL type via moduli problem.

Shimura varieties of PEL type
- P = polarization $\lambda : A \to A^\vee$
- E = endomorphism $B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$
- L = level structure.

Let $F$ be a CM field $\quad * =$ complex conj. on $F \quad \rightsquigarrow F^* := F^{c=1}$

or totally real field. $\quad * = \mathrm{id}$ on $F \quad \rightsquigarrow F^* = F$.

Let $B$ be a simple alg over $\mathbb{Q}$ with center $F$ (e.g. a division algebra over $F$)

$*$ a positive involution on $B$ s.t. $(ab)^* = b^* a^*$ & $*^2 = \mathrm{id}_B$

& $*|_F$ is the $*$ above.

Definition/Lemma $*$ is a positive involution on $B$ if the following equivalent conditions hold

(1) Every f.dim $B_{\mathbb{R}}$-module carries a positive definite Hermitian form :

$$\langle \ , \ \rangle : V \times V \longrightarrow \mathbb{R} \quad \text{symmetric positive definite}$$

$$\& \ \langle bv, w \rangle = \langle v, b^* w \rangle.$$

(2) For every faithful $B$-module $V$, we have $\mathrm{tr}(xx^*, V) > 0$ for $x \in B \setminus \{0\}$.

(3) $\mathrm{Tr}_{B/\mathbb{Q}}(xx^*) > 0$ for $x \in B \setminus \{0\}$

(4) There exists a f.dim $B$-module $V$, s.t. $\mathrm{tr}(xx^*, V) > 0$ for $x \in B \setminus \{0\}$.

(5) There exists a faithful $B$-module $V$ with a positive definite Hermitian form.

Let $V$ be a finite $B$-module, equipped with a non-degenerate $*$-Hermition alternating form.

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{Q}$$

s.t. $\langle bv, w \rangle = \langle v, b^* w \rangle$ , and $\langle v, w \rangle = -\langle w, v \rangle$

Example 1 : $F =$ CM field, $F^+ =$ totally real subfield.

$B = F \curvearrowright V$ an $n$-dim'l $F$-vector space. $V = F^{\oplus n}$

Fix $\delta \in F^{*=-1}$ totally imaginary element. $n = r+s$

$\langle -,- \rangle : V \times V \longrightarrow \mathbb{Q}$

$$\langle (a_1,\dots,a_n),(b_1,\dots,b_n) \rangle = \mathrm{Tr}_{F/\mathbb{Q}}\left(\delta\cdot(a_1\bar{b}_1+\dots+a_r\bar{b}_r - a_{r+1}\bar{b}_{r+1}-\dots-a_n\bar{b}_n)\right)$$

Example 2 $F$ = CM field, $F^+$ = totally real subfield.

$B$ = quaternion division alg $/F$, $\forall b$, $b^* := \overline{(\mathrm{Tr}(b)-b)}$

$V = B$ as left $B$-module. Fix $\delta \in B^{*=-1}$

$\langle -,- \rangle : V \times V \longrightarrow \mathbb{Q}$

$$\langle x_1, x_2 \rangle := \mathrm{Tr}_{B/\mathbb{Q}}(x_1 \delta x_2^*)$$

Check that: $\langle bx_1, x_2 \rangle = \mathrm{Tr}_{B/\mathbb{Q}}(bx_1\delta x_2^*)$, $\langle x_1, b^* x_2 \rangle = \mathrm{Tr}_{B/\mathbb{Q}}(x_1 \delta x_2^* b^{**})$

$$\langle x_1, x_2 \rangle = \mathrm{Tr}_{B/\mathbb{Q}}(x_1\delta x_2^*) = \mathrm{Tr}_{B/\mathbb{Q}}((x_1\delta x_2^*)^*)$$

$$= \mathrm{Tr}_{B/\mathbb{Q}}(x_2 \underbrace{\delta^*}_{-\delta} x_1^*) = -\langle x_2, x_1 \rangle.$$

Set $C := \mathrm{End}_B(V)$. Then the pairing $\langle -,- \rangle$ defines an involution $*$ on $C$:

for $c \in C$, $c^* \in C$ is defined by requiring

$\forall v \in V$, $\langle cv, w \rangle = \langle v, c^* w \rangle$

Check that $c^* \in C$, i.e. $\langle v, c^* bw \rangle \overset{?}{=} \langle v, bc^* w \rangle$

$$\begin{array}{ccc} \langle v, c^* bw \rangle & \overset{?}{=} & \langle v, bc^* w \rangle \\ \| & & \| \\ \langle cv, bw \rangle & & \langle b^* v, c^* w \rangle \\ \| & & \| \\ \langle b^* cv, w \rangle & = & \langle cb^* v, w \rangle \end{array}$$

group for the Shimura varieties

Define the symmilitude group $G$ and its subgroup $G_0$; for $\mathbb{Q}$-algebra $R$

$$G(R) = \{ (g,\lambda) \in (C \otimes_{\mathbb{Q}} R)^\times \times R^\times ;\ g^* g = \lambda \}$$

i.e. $\langle gv, gw \rangle = \lambda \cdot \langle v, w \rangle$

$$\cup\!\!\shortmid$$

$$G_0(R) = \{ g \in (C \otimes_{\mathbb{Q}} R)^\times ;\ g^* g = 1 \}$$

$$\rightsquigarrow \quad 1 \to G_0 \to G \xrightarrow{\lambda} \mathbb{G}_m \to 1$$

Example 1. $C = \mathrm{End}_F(V) = M_n(F)$

$\langle gv, gw \rangle = \lambda \langle v,w \rangle \iff g$ preserves the obvious hermitian form

$((a_1,\cdots,a_n),(b_1,\cdots,b_n)) = a_1\bar{b}_1+\cdots+a_r\bar{b}_r - a_{r+1}\bar{b}_{r+1} - \cdots - a_n\bar{b}_n \in F$

$\Rightarrow G = GU(r,s)$

Example 2. $C = \mathrm{End}_B(V) = \mathrm{End}_B(\underset{\substack{\uparrow \\ \text{left-}B\text{-mod}}}{B_\ell}) \cong B$ ← acting on the right.

In this case $*$ on $C$ is the same as $*$ on $B$

as $\langle cv, w \rangle = \langle v, c^{(*)} w \rangle$ ← the element that we want to show to be equal to $c^*$

$\langle cv,w\rangle = \mathrm{Tr}_{B/\mathbb{Q}}(cv\delta w^*) = \mathrm{Tr}_{B/\mathbb{Q}}(v\delta w^* c)$, $\quad \langle v, c^{(*)}w\rangle = \mathrm{Tr}_{B/\mathbb{Q}}(v\delta(c^{(*)}w)^*)$

$\Big\} \Rightarrow (c^{(*)}w)^* = w^* c$, $\quad c^{(*)} = c^*$

* Special case : $B = B^+ \otimes_{F^+} F$ for a quaternion algebra $/F^+$.

In this case, $G = ((B^+)^\times \times_{F^{+\times}} F^\times)^{\det \in \mathbb{Q}^\times}$

$= \{ (b,e) \in (B^+)^\times \times F^\times \,;\, \det b \cdot \mathrm{Nm}_{F/F^+}(e) \in \mathbb{Q}^\times \} / \{(a,a^{-1}) \,;\, a \in F^{+,\times}\}$

Consider a homomorphism $h : \mathbb{C} \to C_{\mathbb{R}}$ s.t. $h(\bar{z}) = h(z)^*$.

$\rightsquigarrow h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \longrightarrow G_{\mathbb{R}}$.

We require in addition that $\langle v, h(i) w \rangle$ is symmetric positive definite.

$h(z) \in G(\mathbb{C}) \curvearrowright V_{\mathbb{C}} \simeq \underset{\substack{\uparrow \\ \text{acts by } \cdot z}}{V_1} \oplus \underset{\substack{\uparrow \\ \text{acts by } \cdot \bar{z}}}{V_2}$

$E \subseteq \mathbb{C}$ reflex field & $V_1$ as an isomorphism classes of $B$-module is def'd over $E$.

* Let $p$ be a prime number. Will introduce a $p$-unramified version.

Assume the following:

- $F$ is unramified @ $p$.
- $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ = direct product of matrix algs $M_m(\mathbb{Q}_{p^{f_i}})$ ← unram. ext'n of $\mathbb{Q}_p$
- There exists an $\mathcal{O}_{F,(p)}$-subalgebra $\mathcal{O}_B$ of $B$ stable under $*$.
  s.t. $\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ = direct product of matrix algs $M_m(\mathbb{Z}_{p^{f_i}})$

• There exists an $\mathcal{O}_B$-lattice $\Lambda$ of $V$ $(\Lambda / \mathbb{Z}_{(p)})$

s.t. $\langle\, ,\, \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Q}$ takes values in $\mathbb{Z}_{(p)}$

& $\Lambda \simeq \Lambda^\vee$.

* For a scheme $S$, define $AV_S^{(p)}$ to be the category of abelian varieties up to prime-to-$p$ isogenies

Objects: abelian schemes $/S$

Morphisms $\mathrm{Hom}_{AV_S^{(p)}}(A,B) := \mathrm{Hom}_S(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

A prime-to-p polarization on $A$ is a morphism $\varphi : A \longrightarrow A^\vee$ in $AV_S^{(p)}$.

s.t. $\exists\, n \in \mathbb{N}, p \nmid n$ s.t. $n\varphi : A \longrightarrow A^\vee$ is a genuine polarization

& $p \nmid \deg(n\varphi)$.

For $A \in AV_S^{(p)}$, we define $\hat{V}^{(p)}(A) := \left(\varprojlim_{p \nmid n} A[n]\right) \otimes_{\hat{\mathbb{Z}}^{(p)}} \mathbb{A}_f^{(p)}$.

$\Rightarrow$ a morphism $A \to B$ in $AV_S^{(p)}$ gives a map

$\hat{V}^{(p)}(A) \longrightarrow \hat{V}^{(p)}(B)$.

Fix $K^{(p)} \subseteq G(\mathbb{A}_f^{(p)})$ an open compact subgroup, sufficiently small

Theorem. The following functor $\mathcal{M}_{K^{(p)}} : \mathrm{Sch}^{\mathrm{loc.noe}}_{/\mathcal{O}_{E,(p)}} \longrightarrow \mathrm{Sets}$

$S \longmapsto \mathcal{M}_{K^{(p)}}(S) = \Big\{ (A, \lambda, \eta)$

• $A$ is an object in $AV_S^{(p)}$ equipped with a faithful $\mathcal{O}_B$-action satisfies Kottwitz signature condition.

• $\lambda : A \longrightarrow A^\vee$ is a prime-to-$p$ polarization in $AV_S^{(p)}$ s.t. the Rosati involution induces $*$ on $\mathcal{O}_D$.

• a $K^{(p)}$-level structure $i$ :

on each conn. comp. of $S$, choosing a geometric point $\bar{s}$

a $\pi_1(S, \bar{s})$-stable of $K^{(p)}$-orbit of $\mathcal{O}_B$-linear isoms

$$\dim_{\mathbb{Q}} V = \operatorname{rank} H_1^{dR}(A/S)$$

$$\Uparrow$$

$$\underline{\dim_{\mathbb{Q}} V = 2\dim(A/S)}$$

$$\eta_{\bar{s}} : V \otimes_{\mathbb{Q}} \mathbb{A}_f^{(p)} \xrightarrow{\simeq} \hat{V}^{(p)}(A)$$

s.t.

$$\begin{array}{ccccc} V_{\mathbb{A}_f^{(p)}} & \times & V_{\mathbb{A}_f^{(p)}} & \longrightarrow & \mathbb{A}_f^{(p)} \\ \downarrow \eta_{\bar{s}} & & \downarrow \eta_{\bar{s}} & & \wr\!\! \mid \text{ for some fixed elt } c_{\bar{s}} \in \mathbb{A}_f^{(p),\times} \\ \hat{V}^{(p)}(A) & \times & \hat{V}^{(p)}(A) & \xrightarrow{\text{Weil pairing}} & \mathbb{A}_f^{(p)}(1) \end{array}$$

Say $(A, \lambda, \eta)$ and $(A', \lambda', \eta')$ are equivalent if

$\exists$ an $\mathcal{O}_D$-linear morphism $c : A \longrightarrow A'$ in $AV_S^{(p)}$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^\vee \\ c\downarrow & & \uparrow c^\vee \\ A' & \xrightarrow{\lambda'} & A'^\vee \end{array} \quad \text{and} \quad \begin{array}{ccc} & & \hat{V}^{(p)}(A) \\ & \overset{\eta_{\bar{s}}}{\nearrow} & \\ \Lambda_{\mathbb{A}_f^{(p)}} & & \downarrow c_* \\ & \underset{\eta'_{\bar{s}}}{\searrow} & \\ & & \hat{V}^{(p)}(A') \end{array}$$

<u>Kottwitz condition</u>: Let $\alpha_1, \cdots, \alpha_r$ be a $\mathbb{Z}_{(p)}$-basis of $\mathcal{O}_B$. We require

$$\det\big(\alpha_1 X_1 + \cdots + \alpha_r X_r \,;\, \mathrm{Lie}(A/S)\big) \in \mathcal{O}_S[X_1, \cdots, X_r]$$

$$\|$$

$$\det\big(\alpha_1 X_1 + \cdots + \alpha_r X_r \,;\, V_1\big) \in \mathbb{C}[X_1, \cdots, X_r]$$

$$\in \quad \cup\!\mid \qquad\qquad$$

$$\mathcal{O}_{E,(p)}[X_1, \cdots, X_r]$$

<u>Kottwitz condition in explicit terms</u> (over $\mathcal{O}_{E,p}$)

- $$\begin{array}{ccc} \mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p & \curvearrowright & \Lambda \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \\ \| & & \| \\ \prod_i M_m(\mathbb{Z}_{p^{f_i}}) & \curvearrowright & \prod_i \left(\mathbb{Z}_{p^{f_i}}^{\oplus m}\right)^n \end{array}$$

induce a decomposition

Under the $\mathcal{O}_B$-linear action $h(z)$, $V = V_1 \oplus V_2$ over $\mathcal{O}'_{E_p} \simeq \mathbb{Z}_{p^N}$

we get decomposition $\prod \left(\mathbb{Z}^{\oplus m}_{\cdots}\right)^n \otimes \mathbb{Z}_{\cdots} = \prod \prod \left(\mathbb{Z}^{\oplus m}_{\cdots}\right)^n$

$$\prod_i \left(\mathbb{Z}_{p^{f_i}}\right) \otimes \mathbb{Z}_{p^N} = \prod_i \prod_{\tau: \mathbb{Q}_{p^{f_i}} \to \mathbb{Q}_{p^N}} \left(\mathbb{Z}_{p^N}\right)$$

$$= \underbrace{\prod_i \prod_{\tau: \mathbb{Q}_{p^{f_i}} \to \mathbb{Q}_{p^N}}}_{V_1} \left( \left(\mathbb{Z}_{p^N}^{\oplus m}\right)^{r_{i,\tau}} \oplus \left(\mathbb{Z}_{p^N}^{\oplus m}\right)^{s_{i,\tau}} \right) \qquad r_{i,\tau} + s_{i,\tau} = n$$

Then over $S \times_{\mathcal{O}_{E(p)}} \mathbb{Z}_{p^N} =: S_N$

$$\begin{array}{ccc} \omega_{A^\vee/S_N} & \subseteq & H_1^{dR}(A/S_N) \\ \| & & \| \\ \bigoplus_i \bigoplus_\tau \underbrace{e_{i,\tau} \omega_{A^\vee/S_N}}_{\uparrow \text{ rank } s_{i,\tau}}^{\oplus m} & \subseteq & \bigoplus_i \bigoplus_\tau \underbrace{e_{i,\tau} H_1^{dR}(A/S_N)}_{\uparrow \text{ rank } n}^{\oplus m} \end{array}$$

$$e_{i,\tau} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in \circlearrowleft \prod_i \prod_\tau M_m(\mathbb{Z}_{p^N}) = \mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^N}$$

$E_p$ is the subfield where this condition is defined.

<u>Complex uniformization</u>:

Let $X = G(\mathbb{R})$-conjugacy class of homo. $h: \mathbb{S} \to G_{\mathbb{R}}$.

<u>Hope</u>: $\mathcal{M}_{K^{(p)}}(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K^{(p)} G(\mathbb{Z}_p)$ but not quite correct!

$\Psi$

$(A, \lambda, \eta) \rightsquigarrow$ Betti homology $H_1(A, \mathbb{Z}_{(p)})$, $\lambda$, $\mathcal{O}_B$-linear

Étale homology $H_1^{et}(A, \mathbb{A}_f^{(p)}) = \hat{V}^{(p)}(A)$ Weil pairing $\mathcal{O}_B$-linear

If we can choose an isomorphism

$$\alpha: (\Lambda, \langle\,,\rangle) \overset{\mathcal{O}_B \circlearrowright}{\simeq} (H_1(A(\mathbb{C}), \mathbb{Z}_{(p)}), \lambda)$$

then $$(\Lambda_{\mathbb{A}_f^{(p)}}, \langle\,,\rangle) \xrightarrow[\simeq]{\alpha \otimes \mathbb{A}_f^{(p)}} (H_1(A(\mathbb{C}), \mathbb{A}_f^{(p)}), \lambda) \cong (H_1^{et}(A(\mathbb{C}), \mathbb{A}_f^{(p)}), \text{Weil})$$

$\Uparrow \wr\ \eta$ up to $K^{(p)}$

$\simeq$

$(\Lambda \ldots, \langle\,,\rangle)$

$\sim (\Lambda_{\mathbb{A}_f^{(p)}}, \langle\, , \,\rangle)$

gives an element $g_f \in G(\mathbb{A}_f^{(p)})/K^{(p)}$

$$(\Lambda_{\mathbb{R}}, \langle\, , \,\rangle) \xrightarrow[\sim]{\alpha_{\mathbb{R}}} H_1(A(\mathbb{C}), \mathbb{R}) \Big\} \text{ Hodge structure on } H_1(A(\mathbb{C}), \mathbb{R})$$

$$H_1(A(\mathbb{C}),\mathbb{R}) \to \operatorname{Lie} A$$

$$\rightsquigarrow h : \mathbb{S} \longrightarrow GL(H_1(A(\mathbb{C}), \mathbb{R})) \supseteq G_{\mathbb{R}}, \quad h : \mathbb{S} \to G_{\mathbb{R}}$$

Another choice of $\alpha \rightsquigarrow$ change $(g_f, h)$ by an element of $G(\mathbb{Z}_{(p)})$

$$\rightsquigarrow {}_{G(\mathbb{Z}_{(p)})}\backslash X \times G(\mathbb{A}_f^{(p)}) / K^{(p)} \underset{\uparrow}{\simeq} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K^{(p)} G(\mathbb{Z}_p)$$

weak approx. $G(\mathbb{Q})$ dense in $G(\mathbb{Q}_p)$.

One problem however: why does there exist such $\alpha$?

* Note: at each $\ell \neq p$ $\quad \Lambda \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_\ell \xrightarrow[\simeq]{\eta} V_\ell(A)$ compatible with $\lambda$ & $\mathcal{O}_D$

at $\infty$ $\quad$ Kottwitz condition $\Rightarrow \Lambda_{\mathbb{R}} \simeq H_1(A(\mathbb{C}), \mathbb{R})$ comp. w/ $\lambda$ & $\mathcal{O}_D$

at $p$ $\quad$ ✓

$\Rightarrow H_1(A(\mathbb{C}), \mathbb{Q}) \simeq V$ over $\mathbb{A}$ (compatible with $\lambda$ & $\mathcal{O}_D$)

$\rightsquigarrow$ such possible $H_1(A(\mathbb{C}), \mathbb{Q})$ is classified by

$$\operatorname{Ker}\left(H^1(\mathbb{Q}, G) \longrightarrow \prod_v H^1(\mathbb{Q}_v, G)\right) =: Ш^1(\mathbb{Q}, G)$$

<u>Thm</u>: $$\mathcal{M}_{K^{(p)}}(\mathbb{C}) \simeq \coprod_{Ш^1(\mathbb{Q},G)} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K^{(p)} G(\mathbb{Z}_p)$$

<u>Example</u>: unitary case $\quad F = $ CM field. $\quad V$ Herm. space$/F$

$\quad\quad F \,|\, F^+ = $ totally real field

$$G = GU(V)$$

Then $$Ш^1(\mathbb{Q}, G) \cong \begin{cases} \{0\} & \text{if } \dim V = n \text{ even} \\ \operatorname{Ker}\left(F^{+,\times}/\mathbb{Q}^\times \operatorname{Nm}_{F/F^+}(F^\times) \longrightarrow \mathbb{A}_{F^+}^\times / \mathbb{A}^\times \operatorname{Nm}_{F/F^+}(\mathbb{A}_F)\right) \end{cases}$$

if $\dim V = n$ odd

**Note**: it does not matter what the signatures are.

"Proof": $G^{der} = SU(V)$ is simply connected. Will only consider $n$ even

$\uparrow$ derived subgp $[G,G]$.

$G^{ab} = G/G^{der} = $ max'l abelian quotient (not standard notation)

$$(*)\qquad \begin{tikzcd} H^1(\mathbb{Q}, G^{der}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, G) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, G^{ab}) \arrow[d] \\ \prod_v H^1(\mathbb{Q}_v, G^{der}) \arrow[r] & \prod_v H^1(\mathbb{Q}_v, G) \arrow[r] & \prod_v H^1(\mathbb{Q}_v, G^{ab}) \end{tikzcd}$$

Fact: $G^{der}$ simply-connected $\Rightarrow$ $H^1(\mathbb{Q}_v, G^{der}) = \{0\}$ for $v$ nonarch.

$$\Downarrow$$

$$H^1(\mathbb{Q}, G^{der}) \cong H^1(\mathbb{R}, G^{der})$$

* For $GU(V)$, $\quad G \longrightarrow G^{ab} = \{(x,\lambda) \in F^\times \times \mathbb{Q}^\times,\ Nm_{F/F^+}(x) = \lambda^n\}$

$$(g,\lambda) \longmapsto (\det g, \lambda)$$

$$\uparrow$$

$${}^t\bar{g} J g = \lambda J \Rightarrow (\det g)(\det \bar{g}) = \lambda^n$$

· When $n$ even, $\quad G^{ab} \xrightarrow{\sim} U_{F/F^+} \times \mathbb{G}_m$

$$(x,\lambda) \longmapsto (x \cdot \lambda^{-\frac{n}{2}}, \lambda)$$

But $H^1(\mathbb{Q}, U_{F/F^+}) \longrightarrow \prod_v H^1(\mathbb{Q}_v, U_{F/F^+})$ is injective.

e.g. $\quad 1 \to F^{+,\times} \longrightarrow F^\times \longrightarrow U_{F/F^+} \longrightarrow 0$

$$z \longmapsto z/\bar{z}$$

$$\rightsquigarrow \quad \begin{tikzcd} \overbrace{H^1(\mathbb{Q}, \mathbb{G}_{m,F})}^{=0} \arrow[r] \arrow[d] & H^1(\mathbb{Q}, U_{F/F^+}) \arrow[r] \arrow[d, "\text{inj}"] \arrow[r, phantom, shift right=4ex, "\Leftarrow"] & H^2(\mathbb{Q}, F^{+,\times}) \arrow[d, "\text{inj.}"] \\ \prod_v \overbrace{H^1(\mathbb{Q}_v, \mathbb{G}_{m,F})}^{=0} \arrow[r] & \prod_v H^1(\mathbb{Q}_v, U_{F/F^+}) \arrow[r] & \prod_v H^2(\mathbb{Q}, F^{+,\times}) \end{tikzcd}$$

Now $(*)$ becomes $\quad H^1(\mathbb{Q}, G^{der}) \to H^1(\mathbb{Q}, G) \longrightarrow H^1(\mathbb{Q}, G^{ab})$

$$G(\mathbb{R}) \longrightarrow G^{ab}(\mathbb{R}) \xrightarrow{\text{0-map}} H^1(\mathbb{R}, G^{der}) \longrightarrow \prod_v (\mathbb{Q}_v, G) \longrightarrow \prod_v H^1(\mathbb{Q}_v, G^{ab})$$

$$G^{ab}(\mathbb{R}) = \underbrace{U^1(\mathbb{C})}_{\text{Connected}} \times \underbrace{\mathbb{R}^\times}_{\text{2-connected components}}$$

This is surjective b/c it's surj on connected components

$\Rightarrow$ $Ш^1(\mathbb{Q}, G) = \{0\}$ in this case.

In general, if $G^{der}$ is simply-connected, one can prove that

$$Ш^1(\mathbb{Q}, G) \cong Ш^1(\mathbb{Q}, G^{ab})$$

$\downarrow$ will not cover in the lecture

## § Some old writings

* Rosati involution : $b : A \to A$

$$\rightsquigarrow \begin{tikzcd} A \arrow[r, dashed, "b^*"] \arrow[d, "\lambda"] & A \arrow[d, "\lambda"] \\ A^\vee & A^\vee \arrow[l, "b^\vee"'] \end{tikzcd}$$

$\rightsquigarrow$ $*$ is an involution of $B$

$\Rightarrow (b_1 b_2)^* = b_2^* b_1^*$ , $*^2 = id$.

· There's some positivity on $*$ which we will see later.

· Let $F$ be a field. — e.g. $\mathrm{Matrix}_{n\times n}(D)$ $D$ division ring over $F$.

$B$ a semisimple algebra over $F$ – char 0. — e.g. $D = \mathbb{Q}\langle i,j\rangle / (i^2=-a,\ j^2=-b,\ ij=-ji)$

Assume $*$ is an involution of $B$ — such $D$ (say over $F=\mathbb{Q}$), $D \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \simeq \mathrm{Matrix}_{m\times m}(\bar{\mathbb{Q}})$ or product of these

that is an isom $B \xrightarrow{\simeq} B^{op}$ (i.e. $(xy)^* = y^* x^*$) & $*^2 = id$.

<u>Goal</u> : Classify such $(B, *)$ over a number field $F$ (with certain positivity).

First , assume $F$ is algebraically closed.

In this case $F = \prod$ (matrix algebras over $F$)

$\circlearrowleft$ $*$ will permute the factors

2 possible "irreducible types"

- $M = M_n(F) \circlearrowleft *$. Then $b^* = x\,{}^t b\, x^{-1}$ for some $x \in Gl_n(F)$

$$\text{s.t. } b^{**} = x\,{}^t(b^*)\,x^{-1} = x\,{}^t(x\,{}^t b\, x^{-1})\,x^{-1} = x\,{}^t x^{-1}\, b\, {}^t x\, x^{-1}$$

$$\Rightarrow x\,{}^t x^{-1} \in F^\times. \text{ i.e. } {}^t x = c \cdot x \text{ for } c \in F^\times$$

But then $x = {}^{tt}x = c^2 \cdot x \Rightarrow c = \pm 1$

$c = 1$ essentially, $b^* = {}^t b$ Type (C)

$c = -1$ essentially, $b^* = J \cdot {}^t b\, J^{-1}$ for $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ Type (BD)

- $M = M_n(F) \times M_n(F) \circlearrowleft *$ $\quad (x, y) \mapsto ({}^t y, {}^t x)$. Type (A)

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Now, consider the case when $F = \mathbb{R}$,

$(B, *)$ a (fin. dim'l) semisimple $\mathbb{R}$-algebra with $*$ an involution.

<u>Def^n</u>. A Hermitian $B$-module is a finite dim'l $B$-module $V$ with a form $(\,,\,): V \times V \to \mathbb{R}$

s.t. $(bv, w) = (v, b^* w)$

We say that it is <u>positive definite</u> if $(v,v) > 0 \;\; \forall v \in V \setminus \{0\}$.

<u>Easy fact</u>: Positive definite Hermitian $B$-modules are direct sum of irred. ones.

<u>Key Proposition/Definition</u> We say $*$ is a positive involution on $B$ if the following equiv. conds hold.

(1) Every $B$-module carries a positive definite $B$-Herm. form

(5) There's one faithful $B$-module ________________.

(2) For every faithful $B$-module $V$, $\mathrm{tr}(xx^*; V) > 0 \;\; \forall x \in B \setminus \{0\}$

(we don't need faithful here as it is automatic.)

(4) There is one ________________

(3) $\forall x \in B \setminus \{0\}$, $\mathrm{Tr}_{B/\mathbb{R}}(xx^*) > 0$.

Proof: Exercise to prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$

<u>Cor</u>. $(B, *)$ positive involution $\Rightarrow (B^{op}, *)$ positive. (by (3))

$(B_1, *_1), (B_2, *_2)$ positive $\Rightarrow (B_1 \otimes_F B_2, *_1 \otimes *_2)$ positive. (by (3))

$(B,*)$ positive, then any subsemisimple algebra $C$ stable under $*$ $\Rightarrow$ $(C,*)$ positive (by (5))

<u>Cor</u>. $(B,*)$ positive, then $*$ preserves every simple factor of $B$

(o/w, if $*$ interchanges two factors $A_1, A_2$ then for $x \in A_1$, $\mathrm{Tr}_{B/\mathbb{R}}(xx^*) = 0$. $*$)