Lecture 2 of Reading Seminar on Shimura Varieties

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Abstract

In this lecture, we will give introduction to Hilbert modular varieties, unitary Shimura varieties and geometric modular forms.

1 Moduli Problem of Abelian Schemes

Let's picture the general moduli problem of abelian schemes first. The following part can be found in Chap.10 § 2 of [vdG88].

Suppose $\mathscr{B}$ is a scheme and $(\mathcal{P})$ is a property of abelian schemes(e.g. with real multiplication). The moduli problem with respect to $\mathscr{B}$ and $(\mathcal{P})$ is to find a scheme $\mathscr{M}$ over $\mathscr{B}$ with a universal abelian scheme $\mathscr{A}/_{\mathscr{M}}$ such that for any abelian scheme A with property $(\mathcal{P})$ over a base scheme S defined on $\mathscr{B}$, there exists a unique morphism $\phi : S \to \mathscr{M}$ s.t. $A = \mathscr{A} \otimes_\phi S$:

$$\begin{array}{ccc} A & \longrightarrow & \mathscr{A} \\ \downarrow & \ulcorner & \downarrow \\ S & \xrightarrow{\phi} & \mathscr{M} \end{array}$$

In other words, the functor

$$\mathcal{F}_{\mathscr{B},\mathcal{P}} : \qquad Sch/_{\mathscr{B}} \longrightarrow Set$$

$$S \longrightarrow \left\{ \begin{array}{l} \text{Isomorphic classes of abelian} \\ \text{schemes}/_S \text{ with property } \mathcal{P} \end{array} \right\}$$

is represented by $\mathscr{M}$.

Remark 1.1. Sometimes, $\mathcal{P}$ is too weak. $A/_S$ may admit non-trivial automorphisms which implies ϕ can't be unique. Thus we have to strengthen $\mathcal{P}$ to ensure the existence of moduli.

1.1 Hilbert Modular Varieties

Suppose F is a totally real field, i.e. $Hom_{\mathbb{Q}}(\mathbf{F},\mathbb{R}) = Hom_{\mathbb{Q}}(\mathbf{F},\mathbb{C})$, and $\mathfrak{D}$ is the discriminant of F.

Definition 1.2. (Weil Restriction) Let $S' \to S$ be a morphism of schemes. Given an $S'-$scheme X', consider the contravariant functor:

$$Res_{S'/S} : (Sch/_S)^{op} \to Sets$$

sending T to $X'(T \times_S S')$. If the above functor is represented by an $S-$ scheme X, then we say that X is the Weil restriction of X' along $S' \to S$, denoted as $Res_{S'/S}(X')$.

For the case that $S' = Spec(k')$, $S = Spec(k)$, k'/k is a finite extension and X' is an algebraic group, the Weil Restriction exists. For general results and detailed proof for existence of Weil restriction, you may consult [BLR90] or [PR94].

There is no genuine moduli problem for Shimura varieties with respect to $Res_{F/\mathbb{Q}}GL_2$. We will manually define it in the exercises. But we do have a moduli problem of a smaller group $G := (Res_{F/\mathbb{Q}}GL_2)^{\det\in\mathbb{G}_m}$ where, in the view of functor, works as :

$$G(R) = \{g \in GL_2(F \otimes_{\mathbb{Q}} R) | det(g) \in R^\times\}, R \text{ is a } \mathbb{Q}\text{−algebra.}$$

Remark 1.3. G is an algebraic group defined over $\mathbb{Q}$.

Consider the following moduli functor for $N \geq 4$(to make it a scheme)

$$\mathscr{Y}_1(N): \qquad Sch/_{\mathbb{Z}[\frac{1}{N\mathfrak{D}}]} \longrightarrow Sets$$

$$S \longrightarrow \left\{\begin{array}{l}\text{Isomorphic classes of } (A,\lambda,\eta) \text{ such that} \\ \cdot A/_S \text{ abelian scheme of dimension } [F:\mathbb{Q}] \\ \text{equipped with an } \mathcal{O}_F \text{ action;} \\ \cdot\lambda : A \overset{\cong}{\to} A^\vee \text{ is an } \mathcal{O}_F\text{−linear principal} \\ \text{polarization;} \\ \cdot\eta : \underline{\mu_N \otimes_{\mathbb{Z}} \mathcal{O}_F}_{\mu_{n,S}} \hookrightarrow A[N] \text{ is an} \\ \mathcal{O}_F\text{−embedding.}\end{array}\right\}$$

Fact: There are two exact sequences attached to abelian schemes $\pi : A \to S$ in the theory of Shimura varieties:

$$0 \longrightarrow \omega_{A/S} \longrightarrow H^1_{dR}(A/_S) \longrightarrow R^1\pi_*\mathcal{O}_A \longrightarrow 0$$

$$0 \longrightarrow \omega_{A^\vee/S} \longrightarrow H_1^{dR}(A/_S) \longrightarrow Lie_{A/S} \longrightarrow 0$$

where $\omega := \pi_* \Omega^1_{A/S}$ and $H_1^{dR}(A/S) := H^1_{dR}(A^\vee/S)$.

Note that $\mathcal{O}_F-$action respects $S-$scheme structure. Hence it commutes with $\mathcal{O}_S-$action. thus we have $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S - action$ on:

$$0 \longrightarrow \omega_{A^\vee/S} \longrightarrow H_1^{dR}(A/S) \longrightarrow Lie_{A/S} \longrightarrow 0$$

Fact: $H_1^{dR}(A/S)$ is locally free of rank 2 as $\mathcal{O}_F \otimes \mathcal{O}_S-$module.

The principal polarization λ induces a natural alternating perfect pairing:

$$\begin{array}{ccccc} H_1^{dR}(A/S) \times H_1^{dR}(A/S) & \longrightarrow & H_1^{dR}(A/S) \times H_1^{dR}(A^\vee/S) & \longrightarrow & \mathcal{O}_S \\ \big\uparrow & & & & \\ \omega_{A^\vee/S} \times \omega_{A^\vee/S} & & & & \end{array}$$

Fact: $\omega_{A^\vee/S}$ is the annihilator of itself through the above perfect pairing which implies $\omega_{A^\vee/S}$ is locally free of rank 1 as $\mathcal{O}_F \otimes \mathcal{O}_S-$module.

Theorem 1.4. $\mathscr{Y}_1(N)(N \geq 4)$ *is represented by a smooth variety of dimension* $[F : \mathbb{Q}]$ *over* $Spec(\mathbb{Z}[\frac{1}{N\mathfrak{D}}])$ *whose complex points are*

$$G(\mathbb{Q}) \setminus ((\mathfrak{h}^{+[F:\mathbb{Q}]} \sqcup \mathfrak{h}^{-[F:\mathbb{Q}]}) \times (G(\mathbb{A}_f)/\hat{\Gamma}_1(N))).$$

1.2 Unitary Shimura Varieties

Suppose E is an imaginary quadratic field and V is an $E-$vector space equipped with a non-degenerate Hermitian form:

$$< \cdot, \cdot >: V \times V \to E$$

such that

$$< x, y >= \overline{< y, x >}, < ax, by >= a\bar{b} < x, y >, \forall x, y \in V, a, b \in E.$$

Denote $\mathbb{R}i \cap E$ by $E^{c=-1}$. Take $\delta \in E^{c=-1} \setminus 0$. Then δ determines an embedding $E \hookrightarrow \mathbb{C}$ such that $\delta \in \mathbb{R}_{>0} i$.

In the moduli problem for unitary Shimura varieties, we want to link Weil pairing with our hermitian form. However, the Weil pairing doesn't admit any unitary structure. Thus to define the moduli problem, we have to modify the unitary structure.

In fact, $< \cdot, \cdot >$ induces an alternating form:

$$\{\cdot, \cdot\} \to \mathbb{Q}$$

where $\{x, y\} := Tr_{E/\mathbb{Q}}(\delta < x, y >)$.

Note that we have the following properties of $\{\cdot,\cdot\}$:

$$\begin{aligned}\{x,y\} =& Tr_{E/\mathbb{Q}}(\delta < x,y >)\\ =& Tr_{E/\mathbb{Q}}(\delta \overline{< y,x >})\\ =& Tr_{E/\mathbb{Q}}(-\delta < y,x >)\\ =& -Tr_{E/\mathbb{Q}}(\delta < y,x >)\\ =& -\{y,x\};\\ \{ax,y\} =& Tr_{E/\mathbb{Q}}(\delta < ax,y >)\\ =& Tr_{E/\mathbb{Q}}(\delta < x,\overline{a}y >)\\ =& \{x,\overline{a}y\}, \forall a \in E;\end{aligned}$$

$$\begin{aligned}\{x,y\} = 0, \forall x \in V \Rightarrow & Tr_{E/\mathbb{Q}}(\delta < x,y >) = 0, \forall x \in V\\ \Rightarrow & < V,y > \subset \mathbb{Q}\\ \Rightarrow & < V,y >= 0 \text{ since } < V,y > \text{ is an } E- \text{ vector space}\\ \Rightarrow & y = 0 \text{ since } < \cdot,\cdot > \text{ is a perfect pairing.}\end{aligned}$$

In fact, we have the following bijection:

$$\left\{\begin{array}{l}\text{non-degenrate Hermitian forms}\\ \text{on } V:\\ < \cdot,\cdot >: V \times V \to E\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{alternating perfect forms on } V:\\ \{\cdot,\cdot\}: V \times V \to \mathbb{Q}\\ \text{s.t. } \{ax,y\} = \{x,\overline{a}y\}\end{array}\right\}$$

where the inverse map is:

$$< x,y >:= \frac{1}{2}(\frac{1}{\delta}\{x,y\} + \frac{1}{\delta^2}\{\delta x,y\}).$$

Consider the group $GU(V)/_{\mathbb{Q}}$ which works as functor of points in the following way:

$$GU(V)(S) = \{(g,c) \in GL_S(V \otimes_{\mathbb{Q}} S) \times S^{\times} | \forall x,y \in V, \{gx,gy\} = c\{x,y\}\}$$

for any $\mathbb{Q}-$algebra S.

In fact, $\{gx,gy\} = c\{x,y\}$ is equivalent to $< gx,gy >= c < x,y >$. Hence we get the following exact sequence:

$$1 \to U(V) \to GU(V) \to \mathbb{G}_m \to 1.$$

Before stating the moduli problem, let's pick up the level structure first.

For those non-archimedean places, fix an open compact subgroup $K \subset GU(V)(\mathbb{A}_f)$.

For the archimedean place, we have:

$$V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R} = V \otimes_E (E \otimes_{\mathbb{Q}} \mathbb{R}) \simeq V \otimes_E \mathbb{C}$$

whose signature is (a,b), i.e. admitting a basis to make the Hermitian form as $\begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$. The latter isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R}$ is determined by δ. Note that a group homomorphism arises:

$$h: \mathbb{C}^{\times} \to GL(V_{\mathbb{R}}) = GL_n(\mathbb{C})$$

sending z to $\begin{pmatrix} zI_a & 0 \\ 0 & \overline{z}I_b \end{pmatrix}$, which will give us a Shimura data.

The following definition and properties of Shimura data can be found in [Roh09],[Rot05] and [BC13]

Definition 1.5. (Shimura Datum)

A Shimura datum is a pair (G, D) where G is a $\mathbb{Q}-$reductive group and a $G(\mathbb{R})-$conjugacy class X of homomorphisms $h : \mathbb{C}^\times \to G(\mathbb{R})$ of algebraic groups($\mathbb{C}^\times$ is the $\mathbb{R}-$points $Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ here) satisfying:

1.$Ad(h(i))$ is a Cartan involution on $Lie(G^{ad}(\mathbb{R}))$.

2.Only the characters $1, \frac{z}{\overline{z}}, \frac{\overline{z}}{z}$ occur in the representation of $\mathbb{C}^\times$ through $ad \circ h$ on $Lie(G^{ad}(\mathbb{C}))$, i.e. is of Hodge type $(1, -1) \oplus (0, 0) \oplus (-1, 1)$.

3.G^{ad} admits no $\mathbb{Q}-$factor H such that $H(\mathbb{R})$ is compact.

where G^{ad} denotes $G/\mathcal{Z}(G)$.

Remark 1.6. In fact, the third condition can be removed from Shimura datum, which will be explained in an upcoming version of Prof. Xiao an Prof. Zhu's paper.

In fact, we already have an example of Shimura Datum: $(GSp_{2n}, (h'))$ where $h'(a + bi) = aI_{2n} + bJ$. Note that $(V, \{\cdot, \cdot\})$ is a symplectic space. In $V_{\mathbb{R}}$, we the corresponding J is $\begin{pmatrix} iI_a & 0 \\ 0 & -iI_b \end{pmatrix}$. So we can identify our $(GU(V), h)$ as (GSp_{2n}, h') above. Thus $(GU(V), h)$ is a Shimura datum.

Now let's go back to the moduli characterization where we'll identify E as a subfield of $\mathbb{C}$ by the embedding determined by δ:

$$\mathscr{M}_K : \qquad Sch/_E \longrightarrow Sets$$

$$S \longrightarrow \left\{ \begin{array}{l} (A, i, \lambda, \eta)\text{: up to quasi-isogeny:} \\ A \text{ abelian variety of dimension } n \text{ over } S \text{ satisfying a signature condition;} \\ \cdot i : E \hookrightarrow End(A) \text{ an embedding;} \\ \cdot \lambda : A \to A^\vee \text{ an } \mathcal{O}_E\text{-linear quasi-polarization(turning to a polarization after multiplication of a positive integer) s.t the Rosati involution induces complex conjugation on } \mathcal{O}_E. \\ \cdot \text{ On each connected component of } S\text{, fixing a geometric point } \overline{s}, \eta \text{ is a } \pi_1(S, \overline{s})- \text{ stable } K\text{-orbit of } \mathcal{O}_E- \text{ linear isomorphism:} \\ \qquad \eta : V_{\mathbb{A}_f} := V \otimes_{\mathbb{Q}} \mathbb{A}_f \cong \hat{V}(A) \\ \text{compatible with Weil pairing where } \hat{V}(A) := \hat{T}(A) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{array} \right\}$$

Let's explain some terms above now:

Rosati involution: Given an quasi-endomorphism : $A \to A$, the quasi-polarization induces a quasi-endomorphism $\theta_\lambda = \lambda^{-1} \circ \theta^\vee \circ \theta$(we can take inverse λ inside quasi-isogenies):

$$\begin{array}{ccc} A & \xrightarrow{\theta_\lambda} & A \\ \downarrow \lambda & & \downarrow \lambda \\ A^\vee & \xrightarrow{\theta^\vee} & A^\vee \end{array}$$

Compatibility of Weil pairing: $\begin{array}{ccc} V_{\mathbb{A}_f} \times V_{\mathbb{A}_f} & \xrightarrow{\{\cdot,\cdot\}} & \mathbb{A}_f \\ \downarrow \eta\times\eta & & \downarrow \\ \hat{V}(A) \times \hat{V}(A) & \xrightarrow{Weil\ pairing} & \mathbb{A}_f(1) \end{array}$ where the left vertical is multiplication by some locally constant function on S.

Signature condition:

Take L to be a subfield of $\mathbb{C}$ containing all the embedding of E. In our quadratic case, we may take L to be E itself. Suppose S is defined over L.

Consider the exact sequence with $\mathcal{O}_E \otimes_\mathbb{Z} \mathcal{O}_S-$action:

$$0 \to \omega_{A^\vee/S} \to H_1^{dR}(A/_S) \to Lie_{A/_S} \to 0.$$

Since S is defined over $L/\mathbb{Q}$, we have:

$$\begin{aligned} \mathcal{O}_E \otimes_\mathbb{Z} \mathcal{O}_S &\cong \mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Q} \otimes_\mathbb{Q} \mathcal{O}_S \\ &\cong E \otimes_\mathbb{Q} \mathcal{O}_S. \end{aligned}$$

Suppose E is generated by α over $\mathbb{Q}$. Then we have the characterization:

$$E \cong \mathbb{Q}[x]/(\prod_{\sigma \in Hom(E,\mathbb{C})} (x - \sigma(\alpha))).$$

Since $\mathcal{O}_S$ is an $L-$algebra containing $\sigma(\alpha), \forall \sigma \in Hom(E, \mathbb{C})$, we have the following characterization

$$\begin{aligned} \mathcal{O}_E \otimes_\mathbb{Z} \mathcal{O}_E &\cong E \otimes_\mathbb{Q} \mathcal{O}_S \\ &\cong \mathcal{O}_S[x]/(\prod_{\sigma \in Hom(E,\mathbb{C})} (x - \sigma(\alpha))) \\ &\cong \bigoplus_{\sigma \in Hom(E,\mathbb{C})} \mathcal{O}_S \end{aligned}$$

where the last isomorphism comes from Chinese Remainder Theorem. In fact, if we investigate the above isomorphism carefully, it sends $a \otimes s$ to $(\sigma(a)s)_{\sigma \in Hom(E,\mathbb{C})}$ which is canonical.

By the above identification, we have a decomposition of 1 as $\bigoplus_{\sigma \in Hom(E,\mathbb{C})} (1_\sigma)$. Hence we can decompose an $\mathcal{O}_E \otimes \mathcal{O}_S-$ module M as $\bigoplus_{\sigma \in Hom(E,\mathbb{C})} M_\sigma$ where $M_\sigma = 1_\sigma M$. Note that $\mathcal{O}_E$ acts on M_σ through σ.

In our imaginary quadratic case, we can decompose the exact sequence mentioned above as follows:

$$0 \to \omega_{A^\vee/_S,j} \to H_1^{dR}(A/_S)_j \to Lie_{A/_S,j} \to 0, j = 1, 2$$

which is $\mathcal{O}_E$ linear when $j = 1$ and conjugate linear when $j = 2$. Now we are ready to state the signature condition:

$$rank(Lie_{A/_S,1}) = a, rank(Lie_{A/_S,2}) = b.$$

Remark 1.7. Originally, the quasi-polarization λ defers from isomorphism by finite kernel. Since our homology group is defined over L, the kernel vanishes after tensoring. Thus the quasi-polarization induces a perfect pairing:

$$\lambda : H_1^{dR}(A/_S) \times H_1^{dR}(A/_S) \to \mathcal{O}_S.$$

Moreover, the condition that Rosati involution equals conjugate gives us the following pair under decomposition:

$$\begin{array}{lccc} \lambda: & H_1^{dR}(A/_S)_1 \times H_1^{dR}(A/_S)_2 & \longrightarrow & \mathcal{O}_S \\ & \big\uparrow & & \\ & \omega_{A^\vee/_S,1} \times \omega_{A^\vee/_S,2} & & \end{array}$$

<u>Fact:</u> $\omega_{A^\vee/_S,1}, \omega_{A^\vee/_S,2}$ are exact annihilator of each other.

Remark 1.8. The moduli problem can be characterized as isomorphic classes of abelian schemes instead of considering quasi-isogeny. For details, you are recommended to read the first lecture of [HH20]

Theorem 1.9. *When K is sufficiently small in $GU(V)(\mathbb{A}_f)$, $\mathscr{M}_k$ is represented by a smooth variety of dimension ab over E.*

2 Geometric Modular Forms

For details of this section, you may consult Chapter 9 of Prof. Wen-wei Li's book, [Ans20], [Gor02] or the lecture notes we've uploaded in the Wechat group [Iov09].

Let $N \geq 4$. Consider the modular curve $Y_1(N)$ of level $\Gamma_1(N)$ with natural compactification $X_1(N)$.

In the last lecture, we have linked $Y_1(N)$ with the moduli problem of elliptic curves. There exists a universal elliptic curve $\mathscr{E}$ over it. Recall that we have the following exact sequence:

$$0 \to \omega_{\mathscr{E}/_{Y_1(N)}} \to H^1_{dR}(\mathscr{E}/_{Y_1(N)}) \to Lie_{\mathscr{E}^\vee/_{Y_1(N)}} \to 0.$$

For simplicity, we will denote $\omega_{\mathscr{E}/_{Y_1(N)}}$ as ω. ω will naturally extend to the compactification $X_1(N)$

<u>Fact</u> : $\mathrm{M}_k(\Gamma_1(N)) = H^0(X_1(N), \omega^{\otimes k}$ and $S_k(\Gamma_1(N)) = H^0(X_1(N), \omega^{\otimes k}(-C))$ where C is the divisor given by cusps.

Remark 2.1. Note that every elliptic curve is principal polarized: $\mathscr{E} \cong \mathscr{E}^{\vee}$.

Hence, we have a natural perfect alternating pairing from Weil Pairing :

$$H^1_{dR}(\mathscr{E}/_{Y_1(N)}) \times H^1_{dR}(\mathscr{E}/_{Y_1(N)}) \to \mathcal{O}_{Y_1(N)}.$$

Using the alternating property, we will have natural homomorphism:

$$\wedge^2 H^1_{dR}(\mathscr{E}/_{Y_1(N)}) \to \mathcal{O}_{Y_1(N)}.$$

Note that $H^1_{dR}(\mathscr{E}/_{Y_1(N)})$ is locally free of rank 2 as $\mathcal{O}_{Y_1(N)}-$ module. By Nakayama lemma, we will obtain that the above homomorphism is injective. By perfect property, we will get it's surjective. Hence the above morphism is a canonical isomorphism.

Note that ω is locally free of rank 1 as $\mathcal{O}_{Y_1(N)}-$module. Imitating the foregoing argument, we will get an isomorphism :

$$Lie_{\mathscr{E}^{\vee}/_{Y_1(N)}} = H^1_{dR}(\mathscr{E}/_{Y_1(N)})/\omega = \omega^{-1}.$$

So far, we have modified our Hodge filtration exact sequence in the following form:

$$0 \to \omega \to H^1_{dR}(\mathscr{E}/_{Y_1(N)}) \to \omega^{-1} \to 0.$$

Now we are going to construct Hilbert modular forms.

Fix a totally real field over $\mathbb{Q}$ and $N \geq 4$. Suppose $\Sigma := Hom(F, \mathbb{R})$. A paritious weight is a tuple $\kappa := ((k_\tau)_{\tau \in \Sigma}, w) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}$ such that

$$k_\tau \equiv w \pmod 2$$

for every $\tau \in \Sigma$.

Let L be a subfield containing a Galois closure of F. Originally, we define the moduli problem $\mathscr{Y}_1(N)$ with universal abelian scheme $\mathscr{A}_{1,N}$ over $\mathbb{Z}[\frac{1}{N\mathfrak{D}}]$. Now let's make base change to L for it. Denote $\mathscr{Y}_1(N)_L$ by $\mathscr{Y}$ and $\mathscr{A}_{1,N} \otimes L$ by $\mathscr{A}$.

Then we have a classical exact sequence with an $\mathcal{O}_F \otimes \mathcal{O}_{\mathscr{Y}}-$action:

$$0 \to \omega_{\mathscr{A}/\mathscr{Y}} \to H^1_{dR}(\mathscr{A}/\mathscr{Y}) \to Lie_{\mathscr{A}^{\vee}/\mathscr{Y}} \to 0.$$

For simplicity, we will forget $\mathscr{A}$ and $\mathscr{Y}$ from now on.

Imitating the argument in unitary Shimura varieties, decomposition of the above exact sequence will naturally arise:

$$0 \to \omega_\tau \to H^1_{dR,\tau} \to Lie_\tau, \forall \in \Sigma.$$

Define $\epsilon_\tau := \wedge^2 H^1_{dR,\tau}$ which is locally free of rank 1 over $\mathcal{O}_{\mathscr{Y}}$. In fact, as is shown in classical modular forms, ϵ_τ is isomorphic to $O_{\mathscr{Y}}$, thus free of rank 1.

Define $\omega^{\kappa} := \bigotimes_{\tau \in \Sigma} (\omega_\tau^{k_\tau} \otimes \epsilon_\tau^{\frac{w-k_\tau}{2}})$ The associated space of Hilbert modular forms is $H^0(\mathscr{Y}, \omega^k)$.

Remark 2.2. $\mathscr{Y}$ is not proper. But when $[F : Q] > 1$ and all k_τ are equal, we don't need to worry about cusps since all Hilbert modular forms are cusp forms by Koecher principle.

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