# Unitary and Hilbert modular varieties and geometric Hilbert modular forms

## §1 "Fake" Hilbert modular variety

Let $F$ be a totally real field. $D :=$ discriminant of $F$

Key issue: There's no genuine moduli problem for Shimura variety for $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$

But there's a moduli problem for $G = (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2)^{\det \in \mathbb{G}_m}$.

Explicitly, for a $\mathbb{Q}$-algebra $R$, $G(R) = \left\{ g \in \mathrm{GL}_2(R \otimes_{\mathbb{Q}} F) \;\middle|\; \det g \in R^\times \right\}$

In Exercise, we will discuss how to "manually" define the moduli problem for $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$.

Let $N \geq 4$ be an integer ← so that we get a scheme.

Consider the functor

$$\underbrace{\mathcal{Y}_1(N)}: \mathrm{Sch}_{/\mathbb{Z}[\frac{1}{ND}]} \longrightarrow \mathcal{S}ets$$

means $\Gamma_1(N)$-level structure
more general level structure is a bit tricky.

$$S \longmapsto \mathcal{Y}_1(N)(S) = \left\{ \begin{array}{l} \text{isomorphism classes } (A, \lambda, \eta) \\ \bullet\ A \text{ is an abelian scheme } /S \text{ of } \dim = [F:\mathbb{Q}], \\ \quad \text{equipped with an action of } \mathcal{O}_F \\ \bullet\ \lambda : A \xrightarrow{\sim} A^\vee \text{ an } \mathcal{O}_F\text{-linear polarization} \\ \bullet\ \eta : \underbrace{\mu_{N,S} \otimes_{\mathbb{Z}} \mathcal{O}_F} \hookrightarrow A[N] \text{ is an } \mathcal{O}_F\text{-embedding} \end{array} \right\}$$

non-canonically isomorphic to $\mu_{N,S}^{\oplus [F:\mathbb{Q}]}$

For $A$ an abelian variety over $S$, taking relative cohomology for $\pi : A \rightarrow S$, $e$

$$\begin{array}{c} 0 \to \pi_* \Omega^1_{A/S} \to H^1_{dR}(A/S) \to R^1\pi_* \mathcal{O}_A \to 0 \\ \| \wr \\ e^* \Omega^1_{A/S} \\ \| \\ \omega_{A/S} \end{array}$$

b/c $\Omega^1_{A/S}$ admits a basis of invariant differentials

"Dualize" this:

$$\begin{array}{ccccccccc} & & & & H^1_{dR}(A^\vee/S) & & & & \\ & & & & \| & & & & \\ 0 \to & \omega_{A^\vee/S} & \to & & H_1^{dR}(A/S) & \to & \mathrm{Lie}_{A/S} & \to 0 \\ & \circlearrowleft & & & \circlearrowleft & & \circlearrowleft & \\ & \mathcal{O}_F \otimes \mathcal{O}_S & & & \mathcal{O}_F \otimes \mathcal{O}_S & & \mathcal{O}_F \otimes \mathcal{O}_S & \end{array}$$

Very important exact sequence for Sh. var.

Known: $H_1^{dR}(A/S)$ is a locally free $\mathcal{O}_F \otimes \mathcal{O}_S$-module of rank 2.

The polarization $\lambda$ induces a natural pairing

$$H_1^{dR}(A/S)\times H_1^{dR}(A/S) \xrightarrow[\simeq]{1\times\lambda_*} H_1^{dR}(A/S)\times H_1^{dR}(A^\vee/S) \longrightarrow \mathcal{O}_S$$

we assumed to be an isom.

perfect pairing

<u>Fact</u>: $\omega_{A^\vee/S}$ is itself's exact annihilator, or a Lagrangian submodule of $H_1^{dR}(A/S)$.

$\Rightarrow$ $\omega_{A^\vee/S}$ is locally free $\mathcal{O}_F\otimes\mathcal{O}_S$-module of rank 1.

<u>Theorem</u>. $\mathcal{Y}_1(N)$ is represented by a smooth variety of dim $[F:\mathbb{Q}]$ over Spec $\mathbb{Z}[\frac{1}{ND}]$

## §2 Unitary Shimura variety

Let $E$ be an imaginary quadratic ext'n.

Let $V$ be a Hermitian space of dim $n$ over $E$,

that is, $\langle\cdot,\cdot\rangle : V\times V\longrightarrow E$ non-degenerate Hermitian form

$$\langle x,y\rangle = \overline{\langle y,x\rangle}\ ,\quad \langle ax,by\rangle = a\bar{b}\langle x,y\rangle \quad \text{for } x,y\in V,\ a,b\in E.$$

Fix $\delta\in E^{c=-1}$. This determines an embedding $E\subseteq\mathbb{C}$ s.t. $\delta\in\mathbb{R}_{>0}i$

Then the Hermitian form $\langle\cdot,\cdot\rangle$ induces an alternating form

$$\{\cdot,\cdot\} : V\times V\longrightarrow \mathbb{Q}$$ ← This is $\mathbb{Q}$

$$\{x,y\} := \mathrm{Tr}_{E/\mathbb{Q}}(\delta\cdot\langle x,y\rangle)$$

check $\{x,y\} = -\{y,x\}$

$$\overset{\shortparallel}{\mathrm{Tr}_{E/\mathbb{Q}}}(\delta\cdot\langle x,y\rangle) = \mathrm{Tr}_{E/\mathbb{Q}}(\delta\cdot\overline{\langle y,x\rangle}) \overset{c(\delta)=-\delta}{=} \mathrm{Tr}_{E/\mathbb{Q}}(-\delta\cdot\langle y,x\rangle) = -\{y,x\}$$

<u>Fact</u>: $\left\{\begin{array}{c}\text{non-degenerate Herm. forms}\\ \langle\cdot,\cdot\rangle\end{array}\right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{\begin{array}{l}\text{non-degenerate alternating forms } \{\cdot,\cdot\}: V\times V\to\mathbb{Q}\\ \text{satisfying } \{ax,y\} = \{x,\bar{a}y\}\end{array}\right\}$

<u>Note</u>: This bijection depends on the choice of $\delta$.

Consider group $GU(V)$: for $S$ an $\mathbb{Q}$-algebra

$$GU(V)(S) := \left\{ (g,c)\in GL_S(V\otimes_{\mathbb{Q}} S)\times S^\times \;\middle|\; \begin{array}{l}\forall\, x,y\in V\\ \langle gx,gy\rangle = c\langle x,y\rangle\end{array}\right\}$$

similitude unitary group

$$\Updownarrow$$

$$\{gx,gy\} = c\{x,y\}$$

Have $\quad 1 \to U(V) \to GU(V) \xrightarrow{c} \mathbb{G}_m \to 1$

We are interested in understanding the Shimura variety for $GU(V)$.

Fix an open compact subgroup $K \subseteq GU(V)(\mathbb{A}_f)$

Previously, we've talked about the group, then data at all finite places, now at archimedean place

At $\infty$, $V_{\mathbb{R}}$ has signature $(a,b)$ $\quad n = a+b$

i.e. $\exists$ a basis of $V_{\mathbb{R}}$ s.t. the Herm. form is $\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$

$h: \mathbb{S} \to GL(V_{\mathbb{R}}) = GL_n(\mathbb{C})$ Here $E\otimes\mathbb{R} \simeq \mathbb{C}$ uses the embedding determined by $\delta$

$z \longmapsto \begin{pmatrix} z I_a & \\ & \bar{z} I_b \end{pmatrix}$

Then $\{x, h(i)y\} = \mathrm{Tr}_{E_{\mathbb{R}}/\mathbb{R}}(\delta \cdot \langle x, h(i)y\rangle)$

(for $x,y \in E_{\mathbb{R}}^{\oplus a}$, this is $\mathrm{Tr}_{E_{\mathbb{R}}/\mathbb{R}}(c \cdot i \cdot (-i) \cdot \langle x,y\rangle)$ $\quad c \in \mathbb{R}_{>0}$ $\quad$ conjugate-linear in second factor

so positive definite. Similarly for $E_{\mathbb{R}}^{\oplus b}$.)

<u>Moduli functor</u>: $\quad M_K: \mathrm{Sch}^{\text{loc. noe}}_{/E} \longrightarrow \mathcal{S}ets$

This is $E$ instead; viewed <u>canonically</u> as a subfield of $\mathbb{C}$ using the one given by $\delta$.

$S \longmapsto M_K(S) = \Big\{ (A, i, \lambda, \eta)$ : up to quasi-isogenies

- $A$ abelian variety of dim $n$ over $S$ satisfying a signature condition
- $i: E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ an embedding
- $\lambda: A \to A^\vee$ an $\mathcal{O}_E$-linear polarization (only requiring to be quasi isog.)
  s.t. the Rosati involution induces complex conj on $\mathcal{O}_E$
- On each connected component of $S$, fixing a geom. point $\bar{s}$
  $\eta$ is a $\pi_1(S,\bar{s})$-stable $K$-orbit of $\mathcal{O}_E$-linear isom.

$$\eta: V \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A)$$

s.t.

$$\begin{array}{ccc} V_{\mathbb{A}_f} \times V_{\mathbb{A}_f} & \xrightarrow{\{\cdot,\cdot\}} & \mathbb{A}_f \\ \wr\downarrow\eta \quad \wr\downarrow\eta & \circlearrowright & \wr\downarrow \\ \hat{V}(A) \times \hat{V}(A) & \xrightarrow{\text{Weil pairing}} & \mathbb{A}_f(1) \end{array}$$

given by mult by some loc. const func on $S$

<u>Important remark</u>:
Weil pairing can't see the Hermitian form; so we need to turn the Herm. form $\langle , \rangle$ into a $\mathbb{Q}$-valued symplectic form $\{ , \}$ to compare w/ Weil pairing.
This corresponds to a choice of embedding $GU(V) \hookrightarrow GSp_{2g}$.

Rosati involution: Given an endomorphism $\theta: A \to A$, the polarization induces

$$\begin{tikzcd} A \arrow[r, dashed, "\theta"] \arrow[d, "\lambda", "\simeq"'] & A \arrow[d, "\lambda", "\simeq"'] \\ A^\vee \arrow[r, "\theta^\vee"] & A^\vee \end{tikzcd}$$

$\theta_\lambda$ is the Rosati involution (quasi-isogeny)

b/c we chose $S$ to be an $E$-scheme

Signature condition. $0 \to \omega_{A^\vee/S} \to \mathcal{H}_1^{dR}(A/S) \to \mathrm{Lie}_{A/S} \to 0 \quad \circlearrowright \quad \mathcal{O}_S \otimes \mathcal{O}_E \simeq \mathcal{O}_S \oplus \mathcal{O}_S$

$\mathcal{H}_1^{dR}(A/S) \circlearrowleft \underbrace{\mathcal{O}_E \otimes \mathcal{O}_S}$ $\qquad x \otimes a \longmapsto (ax, \bar{a}x)$

locally free $\mathcal{O}_E \otimes \mathcal{O}_S$-module of rank $n$

According to the decomposition on the right, got decomposition $\quad 0 \to \omega_{A^\vee/S,j} \to \mathcal{H}_1^{dR}(A/S)_j \to \mathrm{Lie}_{A/S,j} \to 0 \quad$ $j=1$. $\mathcal{O}_E$-linear; $j=2$ $\mathcal{O}_E$-conj-linear

Require: $\mathrm{rank}\,(\mathrm{Lie}_{A/S,1}) = a$, $\mathrm{rank}\,(\mathrm{Lie}_{A/S,2}) = b$.

(corresponding to the condition for $h$ earlier.)

Remark: The polarization $\lambda$ induces a perfect pairing

$$\lambda : \mathcal{H}_1^{dR}(A/S) \times \mathcal{H}_1^{dR}(A/S) \longrightarrow \mathcal{O}_S$$

But the Rosati involution condition implies that under the decomposition into $j=1,2$.

$$\text{we get} \quad \lambda : \mathcal{H}_1^{dR}(A/S)_1 \times \mathcal{H}_1^{dR}(A/S)_2 \longrightarrow \mathcal{O}_S$$

$$\begin{array}{ccc} \mathcal{H}_1^{dR}(A/S)_1 & & \mathcal{H}_1^{dR}(A/S)_2 \\ \cup\!\mathrm{l} & & \cup\!\mathrm{l} \\ \omega_{A^\vee/S,1} & & \omega_{A^\vee/S,2} \end{array}$$

rank $n-a=b$ $\qquad\qquad$ rank $n-b=a$

$\omega_{A^\vee/S,1}$ & $\omega_{A^\vee/S,2}$ are exact annihilator of each other.

Theorem When $K \leq GU(V)(\mathbb{A}_f)$ is sufficiently small, $\mathcal{M}_K$ is represented by a smooth variety of dimension $a \cdot b$ over $E$.

Remark: Similar to the argument for modular curve & Siegel moduli variety, can "almost" prove $\mathcal{M}_K(\mathbb{C}) = {}_{G(\mathbb{Q})}\backslash\left(X \times \left(G(\mathbb{A}_f)/K\right)\right)$

$$\text{where} \quad X = GU_{\mathbb{R}}(a,b)(\mathbb{R}) \big/ G(U_{\mathbb{R}}(a) \times U_{\mathbb{R}}(b))(\mathbb{R}) \overset{\text{if } a \neq b.}{=} U_{\mathbb{R}}(a,b) \big/ U_{\mathbb{R}}(a) \times U_{\mathbb{R}}(b)$$

Not quite correct. When $n=a+b$ is even, this is okay.

When $n=a+b$ is odd, $\mathcal{M}_K(\mathbb{C}) =$ finite identical copies of this.

Will explain this subtlety in later lectures.

## §3 Geometric modular forms

Let $N \geq 4$. Consider modular curve $Y_1(N)$ of level $\Gamma_1(N)$.

It has a natural compactification $X_1(N)$. Will discuss this in details in Talk 4.

Will pretend that $Y_1(N)$ is already proper over $\mathbb{Q}$.

Will use the moduli problem $Y_1(N)(S) = \{(E, \iota) \,;\, E \text{ elliptic curve}/S,\ \iota : \mathbb{Z}/N\mathbb{Z}_S \hookrightarrow E[N]\}$

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ Y_1(N) \end{array} \qquad 0 \to \underset{\omega :=}{\omega_{\mathcal{E}/Y_1(N)}} \to H^1_{dR}(\mathcal{E}/Y_1(N)) \to \mathrm{Lie}_{\mathcal{E}^\vee/Y_1(N)} \to 0 \qquad (*)$$

Somehow, the theory of automorphic forms works better with cohomological convention.

To study the geometry, homology is prefered as it is covariant functorial.

There is a natural extension of $\omega$ to the compactification $X_1(N)$.

Fact : $M_k(\Gamma_1(N)) = H^0(X_1(N), \omega^{\otimes k})$ weight $k$ modular forms

can't use $Y_1(N)$ b/c $Y_1(N)$ is not proper

$\qquad \cup\!|$

$S_k(\Gamma_1(N)) = H^0(X_1(N), \omega^{\otimes k}(-C))$ where $C$ = divisor given by cusps.

* Elliptic curves are naturally principally polarized, $\mathcal{E} \cong \mathcal{E}^\vee$.

$\Rightarrow$ natural perfect pairing $H^1_{dR}(\mathcal{E}/Y_1(N)) \times H^1_{dR}(\mathcal{E}/Y_1(N)) \to \mathcal{O}_{Y_1(N)}$

$\Rightarrow$ canonical isomorphism. $\wedge^2 H^1_{dR}(\mathcal{E}/Y_1(N)) \simeq \mathcal{O}_{Y_1(N)}$

So the exact sequence becomes

$$0 \to \omega_{\mathcal{E}/Y_1(N)} \to H^1_{dR}(\mathcal{E}/Y_1(N)) \to \omega^{-1}_{\mathcal{E}/Y_1(N)} \to 0$$

## §4 Geometric Hilbert modular forms

A major issue we are fighting is : moduli problem $\leftrightarrow G' = (\mathrm{Res}_{F/\mathbb{Q}} GL_2)^{\det \in \mathbb{G}_m}$

automorphic application $\leftrightarrow G = GL_2$.

In this talk : We introduce the "correct" definition of geometric HMF on the moduli problem for $G'$.

In the exercise : We explain how to modify the moduli problem to get to the Shimura variety for $G$

& then our definition will "automatically" generalize.

$F$ totally real field over $\mathbb{Q}$. $N \geq 4$.

$\Sigma := \mathrm{Hom}(F, \mathbb{R})$. set of real embeddings

A paritious weight is a tuple $\kappa := ((k_\tau)_{\tau \in \Sigma}, w) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}$ s.t. $k_\tau \equiv w \mod 2$ for every $\tau$.

means satisfying certain property about parity of numbers.

Remark: This is similar to that for $\Gamma_0(N)$-level, we need even weights.

Here for HMFs, the constraint on weights is more serious.

Let $L$ be a subfield of $\mathbb{C}$ containing a Galois closure of $F$

$$\begin{array}{l} \mathcal{A} \quad \text{universal abelian variety} \\ \big| \\ \mathcal{Y}_1(N)_L =: \mathcal{Y} \end{array} \qquad 0 \to \omega_{\mathcal{A}/\mathcal{Y}} \to \mathcal{H}^1_{dR}(\mathcal{A}/\mathcal{Y}) \to \mathrm{Lie}_{\mathcal{A}^\vee/\mathcal{Y}} \to 0$$

$\leftarrow$ base change to $L$.

$$\mathcal{O}_F \otimes \mathcal{O}_\mathcal{Y} \simeq \bigoplus_{\tau \in \mathrm{Hom}(F,L) = \Sigma} \mathcal{O}_\mathcal{Y}$$

decompose $\rightsquigarrow$ $0 \to \underset{\text{rank } 1}{\omega_\tau} \to \underset{\text{rank } 2 \text{ over } \mathcal{O}_\mathcal{Y}}{\mathcal{H}^1_{dR,\tau}} \to \underset{\text{rank } 1}{\mathrm{Lie}_\tau} \to 0$

Define $\mathcal{E}_\tau := \wedge^2 \mathcal{H}^1_{dR,\tau}$ rank 1 over $\mathcal{O}_\mathcal{Y}$

Similar to above, the polarization $\lambda : \mathcal{A} \to \mathcal{A}^\vee$ induces a perfect pairing

$$\lambda : \mathcal{H}^1_{dR}(\mathcal{A}/S) \times \mathcal{H}^1_{dR}(\mathcal{A}/S) \longrightarrow \mathcal{O}_S$$

$$\rightsquigarrow \lambda_\tau : \mathcal{H}^1_{dR,\tau} \times \mathcal{H}^1_{dR,\tau} \longrightarrow \mathcal{O}_S$$

$\Rightarrow$ canonical isomorphism $\wedge^2 \mathcal{H}^1_{dR,\tau} \xrightarrow{\cong} \mathcal{O}_S$ (depending on $\lambda$)

Define $$\omega^\kappa = \bigotimes_{\tau \in \Sigma} \left( \omega_\tau^{k_\tau} \otimes \mathcal{E}_\tau^{\frac{w-k_\tau}{2}} \right)$$

Remark: Indeed, on $\mathcal{Y}_1(N)$ as defined, $\mathcal{E}_\tau$ is canonically trivialized.

But our ultimate goal is to study Hilbert modular forms on $G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$

Need to do some "modification" to this picture, including introducing an action of $u \in \mathcal{O}_F^{\times,>0}$

which acts non-trivially on $\mathcal{E}_\tau$ by mult. by $\tau(u)^2$. & on $\omega_\tau$ by mult. by $\tau(u)$

But totally, the action is mult by $\prod_{\tau \in \Sigma} \left( \tau(u)^{k_\tau} \cdot \tau(u^2)^{\frac{w-k_\tau}{2}} \right) = \mathrm{Nm}_{F/\mathbb{Q}}(u)^w = 1$.

The associated space of Hilbert modular forms is

$$H^0\left(\mathcal{Y}_1(N), \omega^k\right)$$

Remark: We here again ignored the issue that $\mathcal{Y}_1(N)$ is not proper.
But when $[F:\mathbb{Q}]>1$, and all $k_\tau$ are equal, we don't need to worry about the cusps, because all HMFs are automatically convergent by Hartogs' theorem.