Background material on algebraic geometry

§1. Torsors for group schemes

Fix a noetherian scheme S as the base (probably don't need S to be noetherian)

Definition: A group scheme G over S is a scheme G over S $(\pi : G \to S)$

together with
- $m : G \times_S G \longrightarrow G$ multiplication
- $i : G \xrightarrow{\simeq} G$ inverse
- $e : S \overset{\text{closed}}{\hookrightarrow} G$ a section of π (i.e. $\pi \circ e = id$)

satisfying "obvious" axioms:

(1)
$$\begin{tikzcd}
G \times_S G \times_S G \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] \arrow[dr, phantom, "\circlearrowleft"] & G \times_S G \arrow[d, "m"] \\
G \times_S G \arrow[r, "m"] & G
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
S \times_S G \arrow[r, hook, "e \times 1"] \arrow[dr, "id"'] & G \times_S G \arrow[d, "m"] & G \times_S S \arrow[l, hook', "1 \times e"'] \arrow[dl, "id"] \\
& G &
\end{tikzcd}$$
$\circlearrowleft \quad \circlearrowleft$

(3)
$$\begin{tikzcd}
G \arrow[r, "1 \times e"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & G \times_S G \arrow[d, "m"] \\
S \arrow[r, "e"] & G
\end{tikzcd}$$
Cartesian.

Ex: on the level of rings what is m^* for $G = \mathbb{G}_m$,

Ex $\alpha_p, \mu_p, \mathbb{Z}/p\mathbb{Z}, \ldots$

- Say G is a (finite) (flat) group scheme if $G \to S$ is (finite) (flat).

Because of such a definition, for every S-scheme T,

$$G(T) = \mathrm{Mor}_S(T, G) \text{ is a group.}$$

Definition: Let G be a flat group scheme over S. A G-torsor or a G-bundle for the Zariski topology is a scheme $\mathcal{E} \xrightarrow{\pi} S$ s.t.

(1) G acts on $\mathcal{E}$ in the sense that

$\exists$ a morphism $act : G \times_S \mathcal{E} \to \mathcal{E}$ satisfying the "obvious" axioms

(2) $\mathcal{E}$ is locally trivial in the sense that, there's a Zariski covering $\{U_i\}$ of S

$\exists$ an isom. $\mathcal{E} \times_S U \overset{\phi}{\simeq} G \times_S U$ s.t.
$$\begin{tikzcd}
G \times_S \mathcal{E} \times_S U \arrow[r, "act"] \arrow[d, "1 \times \phi", "\wr"'] & \mathcal{E} \times_S U \arrow[d, "\phi", "\wr"']
\end{tikzcd}$$

$$G\times_S G\times_S U \xrightarrow{m} G\times_S U.$$

Easy to see $\{$isom. classes of G-torsors on $S\} \leftrightarrow \check{H}^1(S,G) \leftarrow$ Čech cohomology

$$\begin{tikzcd} \check{H}^1(S,G) \arrow[d, phantom, "\shortparallel"] \\ \varinjlim_{\substack{\{U_i\}\\ \text{affine cover}}} \mathrm{Ker}\left(\prod_{i<j} G(U_i\cap U_j) \to \prod_{i<j<k} G(U_i\cap U_j\cap U_k)\right) \end{tikzcd}$$

In many ocassions, G is over $\operatorname{Spec}\mathbb{Z}$, then a G-torsor over S means a G_S-torsor over S.

__Example__: $\{$vector bundles of rank n over $S\} \xleftrightarrow{\text{bij}} \{GL_n$-torsor over $S\}$

$\leftrightarrow \check{H}^1(S, GL_n(\mathcal{O}_S))$

* Conversely, if G is defined over $\operatorname{Spec}\mathbb{Z}$ and $G \to GL(V)$ is a representation then there is a natural functor (preserving natural tenser & dual)

$$\begin{tikzcd} \{\text{Alg. Rep'ns of } G\} \arrow[r] & \{\text{Vector bundles on } X\} \\ V \arrow[r, mapsto] & \mathcal{G}\overset{G}{\times} V := (\mathcal{G}\times V)/\text{diagonal } G\text{-action} \end{tikzcd}$$

Locally, we make the quotient $(G\times X\times V)/G \cong X\times V$ but there's a global twist.

Or equivalently, $\check{H}^1(X,G) \to \check{H}^1(X, GL(V))$
$[\mathcal{G}] \longmapsto [\mathcal{G}\overset{G}{\times} V].$

Back to the case of unitary Shimura variety M for $G = GU(V)$ of signature (a,b)

$$\begin{tikzcd} A \arrow[d] \arrow[r, squiggly] & \omega_{A/M,1} \text{ has rank } b \arrow[d, phantom, "\wr"] & \text{and} & \omega_{A/M,2} \text{ has rank } a \arrow[d, phantom, "\wr"] \\ M & \mathcal{G}_1 \ GL_b\text{-torsor} & & \mathcal{G}_2 \ GL_a\text{-torsor.} \end{tikzcd}$$

So any irreducible rep'n $V_b\otimes V_a$ of $GL_b\times GL_a \leftrightarrow$ some highest wt of $\underset{\substack{\cap \\ GL_n}}{GL_b\times GL_a}$

$\rightsquigarrow \left(\mathcal{G}_1 \overset{GL_b}{\times} V_b\right) \otimes \left(\mathcal{G}_2 \overset{GL_a}{\times} V_a\right)$ an automorphic vector bundle on M.

Sections are automorphic forms.

§2. Gauss-Manin connections

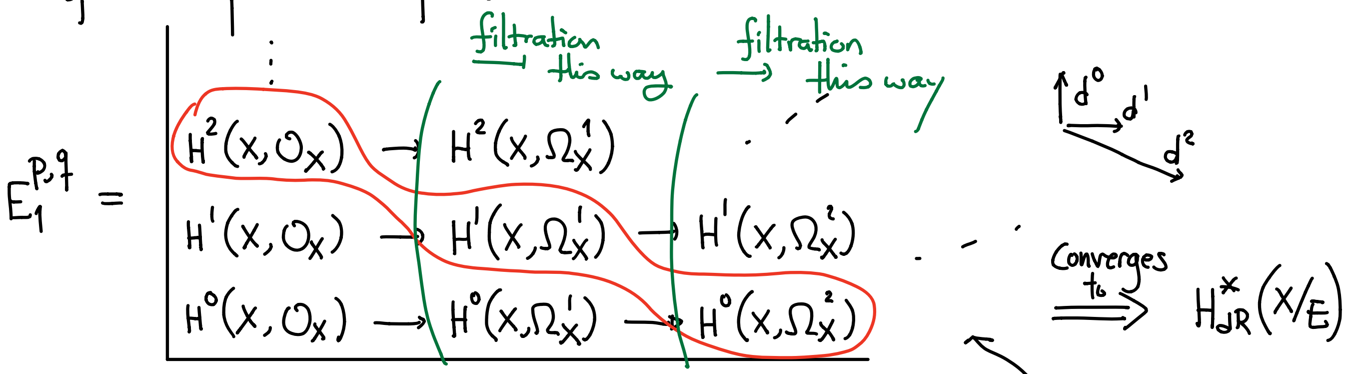
$$\begin{tikzcd} X \arrow[d] \\ \operatorname{Spec} E \end{tikzcd}$$

X proper smooth variety. Note: We emphasize that X is defined over a subfield $E \subseteq \mathbb{C}$ b/c the de Rham cohomology $H^n_{dR}(X/E)$ is canonically defined$/E \subseteq \mathbb{C}$

$\operatorname{Spec} E$, $E \subseteq \mathbb{C}$ de Rham cohomology $H^n_{dR}(X/E) := \mathbb{H}^n(X, \Omega^\bullet_{X/E})$

Here $\Omega^\bullet_{X/E} = \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/E} \xrightarrow{d} \Omega^2_{X/E} := \wedge^2 \Omega^1_{X/E} \xrightarrow{d} \cdots$ is the de Rham complex

⇝ spectral sequence to compute it:



$$\begin{tikzcd} \vdots & & & \\ H^2(X,\mathcal{O}_X) \arrow[r] & H^2(X,\Omega^1_X) & & \\ H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\Omega^1_X) \arrow[r] & H^1(X,\Omega^2_X) & \cdots \\ H^0(X,\mathcal{O}_X) \arrow[r] & H^0(X,\Omega^1_X) \arrow[r] & H^0(X,\Omega^2_X) & \end{tikzcd}$$

$E_1^{p,q} =$ (grid above) $\xRightarrow{\text{Converges to}} H^*_{dR}(X/E)$

Fact: This spectral sequence degenerates at E_1, i.e. all maps d^1 are zero

But there are additional information, e.g. look at $H^2_{dR}(X/E)$

⇝ get Hodge filtration $\underset{\overset{\shortparallel}{H^0(X,\Omega^2_X)}}{F^2 H^2_{dR}(X/E)} \subseteq F^1 H^2_{dR}(X/E) \subseteq F^0 H^2_{dR}(X/E) = H^2_{dR}(X/E)$

↖ subquot is $H^1(X,\Omega^1_X)$ ↖ subquot is $H^2(X,\mathcal{O}_X)$ decreasing filtration!

In general, we write

$$H^n_{dR}(X/E) = \left(H^0(X,\Omega^n_X) - H^1(X,\Omega^{n-1}_X) - H^2(X,\Omega^{n-2}_X) - \cdots - H^n(X,\mathcal{O}_X) \right)$$

de Rham comparison $H^n(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes \mathbb{C} \simeq H^n_{dR}(X/E) \otimes_E \mathbb{C}$

$$\xrightarrow{\text{by Hodge theory}} \cong \bigoplus_{p+q=n} H^{p,q}(X(\mathbb{C}))$$

We can alternatively characterize $H^{p,q}(X(\mathbb{C}))$ as $F^p H^n_{dR}(X/\mathbb{C}) \cap \overline{F^q H^n_{dR}(X/\mathbb{C})}$

Now, let X vary in family S is a smooth E-scheme. E a given subfield of $\mathbb{C}$

$$\begin{tikzcd} X \arrow[d, "\pi"', "\text{proper smooth.}"] \\ S \end{tikzcd}$$

$S/\operatorname{Spec} E$

⇝ $\Omega^\bullet_{X/S}: \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} := \wedge^2 \Omega^1_{X/S} \xrightarrow{d} \cdots \longrightarrow \Omega^{\dim_S X}_{X/S}$

$H^\bullet_{dR}(X/S) := R\pi_*(\Omega^\bullet_{X/S})$

comes from

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\Omega^1_S \arrow[r] & \Omega^1_X \arrow[r] & \Omega^1_{X/S} \arrow[r] & 0 \\ & & \mathcal{O}_X \arrow[u, "d"'] & & \end{tikzcd}$$

There's a family of Hodge filtration

$$H^n_{dR}(X/S) = \left(\pi_* \Omega^n_{X/S} - R^1\pi_* \Omega^{n-1}_{X/S} - R^2\pi_* \Omega^{n-2}_{X/S} - \cdots - R^n \pi_* \mathcal{O}_X \right)$$

Fact: $\mathcal{H}^n_{dR}(X/S)$ is a vector bundle over S, equipped with a Gauss-Manin connection

$$\nabla_{GM}: \mathcal{H}^n_{dR}(X/S) \longrightarrow \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E}$$

$$\nabla_{GM}(a \cdot x) = x \otimes da + a \cdot \nabla(x) \quad \text{for } a \in \mathcal{O}_S,\ x \text{ a section of } \mathcal{H}^n_{dR}(X/S)$$

(1) The Gauss-Manin connection is integrable:

$$\mathcal{H}^n_{dR}(X/S) \xrightarrow{\nabla_{GM}} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E} \xrightarrow{\nabla_{GM}} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^2_{S/E}$$

$$x \otimes \xi \longmapsto \nabla_{GM}(x) \wedge \xi + x \otimes d\xi$$

satisfies $\nabla^2_{GM} = 0$

, so that $\left(\mathcal{H}^n_{dR}(X/S) \otimes \Omega^\bullet_{S/E}, \nabla_{GM}\right)$ is a complex of sheaves on S.

Rmk: If S is also proper & smooth, then

$$H^\bullet_{dR}(X/E) \simeq H^\bullet_{dR}\left(S, \mathcal{H}^\bullet_{dR}(X/S) \otimes \Omega^\bullet_{S/E}\right)$$

(2) ∇_{GM} does not preserves the natural filtration $F^\bullet \mathcal{H}^n_{dR}(X/S)$. but "almost"

Griffith transversality $\quad \nabla_{GM}\left(\mathrm{Fil}^i \mathcal{H}^n_{dR}(X/S)\right) \subseteq \mathrm{Fil}^{i-1} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E}$

e.g.

$$\begin{tikzcd}
 & \pi_* \Omega^2_{X/S} \arrow[d, dash] & \pi_* \Omega^2_{X/S} \otimes \Omega^1_{S/E} \arrow[d, dash] & \\
\mathcal{H}^2_{dR}(X/S) = & R^1\pi_* \Omega^1_{X/S} \arrow[d, dash] \arrow[r, "\nabla_{GM}"] & R^1\pi_* \Omega^1_{X/S} \otimes \Omega^1_{S/E} \arrow[d, dash] & = \mathcal{H}^2_{dR}(X/S) \otimes \Omega^1_{S/E} \\
 & R^2\pi_* \mathcal{O}_X & R^2\pi_* \mathcal{O}_X \otimes \Omega^1_{S/E} &
\end{tikzcd}$$

Remark: Maybe the "correct" formulation is to put $\Omega^1_{S/E}$ in fil^1 so that ∇_{GM} now preserves the Hodge filtration.

Cor: ∇_{GM} is not a morphism of coherent sheaves (it's more like a differential operator)

But ∇_{GM} induces on a morphism of coherent sheaves on graded piece:

$$\begin{array}{ccc}
\mathrm{gr}^i \nabla_{GM}: \mathrm{gr}^i \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \mathrm{gr}^{i-1} \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E} \\
\| & & \| \\
R^i \pi_* \Omega^{n-i}_{X/S} & & R^{i-1} \pi_* \Omega^{n-i+1}_{X/S}
\end{array}$$

Proof: if $a \in \mathcal{O}_S$, $x \in F^i \mathcal{H}^n_{dR}(X/S)$, $\nabla_{GM}(a \cdot x) = \underbrace{x \otimes da}_{\underset{F^i \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E}}{\cap}} + a \cdot \nabla_{GM}(x)$

$F^i \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/E}$, so 0 in $\mathrm{gr}^{i-1} \mathcal{H}^n_{dR}(X/S) \otimes \Omega_S$

$\Rightarrow \mathrm{gr}^i \nabla_{GM}$ is a morphism of coherent sheaves.

Over $\mathbb{C}$, by which we meant $X_{\mathbb{C}} := X \otimes_E \mathbb{C}$, $S_{\mathbb{C}} := S \otimes_E \mathbb{C}$

we have Betti cohomology $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q}) := R^n \pi_* \underline{\mathbb{Q}}_{X^{an}_{\mathbb{C}}}$

Betti-de Rham comparison: $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\underline{\mathbb{Q}}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}} \xrightarrow{\sim} \mathcal{H}^n_{dR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}}$

$$(1 \otimes \nabla_{S^{an}_{\mathbb{C}}}) \longleftrightarrow \nabla_{GM}$$

In particular, all sections of $\mathcal{H}^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{Q})$ are horizontal

§ 3 Hodge structure and its variation

Let $R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$, a subring of $\mathbb{R}$.

imagine to be $H^n(X(\mathbb{C})^{an}, R)$

A R-Hodge structure is a finite projective R-module V, equipped with a bigrading

$$V_{\mathbb{C}} := V \otimes_R \mathbb{C} \simeq \bigoplus_{p,q} H^{pq}$$

s.t. $H^{p,q} = \overline{H^{q,p}}$ (for the complex conjugation on $\mathbb{C}$: $V \otimes_R \mathbb{C} \longrightarrow V \otimes_R \mathbb{C}$)

$$v \otimes z \longmapsto v \otimes \bar{z}$$

The pairs (p,q) for which $H^{p,q} \neq 0$ are called types of V, counted with mult. $\dim_{\mathbb{C}} H^{pq}$

Say the Hodge structure is of pure wt n if for all types (p,q) of V, $n = p+q$.

In this case, the bigrading decomposition is the same as the filtration

$$F^p V := \bigoplus_{r \geq p} V^{rs} \quad , \text{ as } V^{pq} = F^p V \cap \overline{F^q V}.$$

Write $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, so that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and

$$\mathbb{S}(\mathbb{C}) \cong \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$$

$$z \otimes r \longmapsto (zr, \bar{z}r)$$

must be linear in coeff r.

Exercise: Giving an R-Hodge structure on V is equivalent to give a homomorphism

$$h : \mathbb{S} \longrightarrow GL(V_{\mathbb{R}}) \quad \text{s.t.}$$

$$\begin{array}{ccc} h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \longrightarrow & GL(V_{\mathbb{C}}) \\ & & \parallel \\ (z_1, z_2) & & \bigoplus H^{p,q} \ni v^{pq} \end{array}$$

$$h(z_1, z_2) v^{pq} = z_1^{-p} z_2^{-q} v^{pq}$$

Rmk: Complex conjugation descends $h_{\mathbb{C}}$ down to h.

Let S be a complex analytic variety over k

Definition A variation of Hodge structure (VHS) on S of pure weight n consists of

- a local system of projective R-module V on S.

  (that is a locally free sheaf for analytic topology of projective R-mods)
- a decreasing filtration $F^p\mathcal{V}$ on the vector space $\mathcal{V} := V\otimes\mathcal{O}_S$ satisfying the transversality

  (note : ($\mathcal{V} = V\otimes\mathcal{O}_S$, $1\otimes\nabla$) is a natural vectorbundle with integrable connection,

  we require $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V}\otimes\Omega^1_S$
- for every $s\in S$, the filtration induces on V_s a pure Hodge structure of weight n.

A polarization of a VHS is a perfect bilinear pairing

$$\psi : V\times V \longrightarrow \underline{R}_S$$

← const sheaf over S

s.t. $\Psi_{\mathbb{R}}(x, h(i)y)$ is symmetric and positive definite on every fiber of S.

Typical example comes from

$$\begin{tikzcd} X \arrow[d, "\pi"] & \text{projectivesmooth} \\ S & \end{tikzcd}$$

$$V = R^n\pi^{an}_*\mathbb{Q}_X = \mathcal{H}^n_B(X/S, \mathbb{Q})$$

and $\mathcal{V} = V\otimes\mathcal{O}_S \cong \mathcal{H}^n_{dR}(X/S)$ carrying a Hodge filtration.

To see the polarization, we embed

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}^N \arrow[dl, "f"] \\ S & \end{tikzcd}$$

Then the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ restricts to $\mathcal{O}_X(1)$ & defines a class

$$c \in H^0(S, R^2\pi^{an}_*\mathbb{Q})$$

By Hard-Lefschetz, $R^n\pi^{an}_*\mathbb{Q} \xrightarrow{\cong} R^{2d-n}f_*\mathbb{Q}$

$$x \longmapsto c^{d-n}\wedge x \qquad (\text{if } d\geq n)$$

Then Poincaré duality gives V a natural polarization

$$R^n\pi^{an}_*\mathbb{Q}\times R^n\pi^{an}_*\mathbb{Q} \xrightarrow{1\times c^{d-n}} R^n\pi^{an}_*\mathbb{Q}\times R^{2d-n}\pi^{an}_*\mathbb{Q} \xrightarrow{\cup} R^{2d}\pi^{an}_*\mathbb{Q} \cong \underline{\mathbb{Q}}_S.$$