Lecture 4 of Reading Seminar on Shimura Varieties

姜杰东/Jiang Jiedong

May 21, 2020

1 Katz's interpretation of modular forms

We use $X_1(N)$, $N \geq 4$ as an example to explain Katz's interpretation. Recall the definition of $X_1(N)$ and the modular form of weight k on it.

Definition 1.1. $X_1(N)$ represents the following functor:

$$\begin{aligned}\mathcal{M}_1(N) : \mathbf{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathbf{Set} \\ S &\mapsto \mathcal{M}_1(N)(S) = \left\{ \begin{array}{l} \text{Isomorphism classes of } (E, i) \text{, where} \\ E \text{ is a "generalized" elliptic curves over } S, \\ i : \mu_{N/S} \hookrightarrow E[N] \text{ is a level } N \text{ structure.} \end{array} \right\}\end{aligned}$$

A *weight k modular form* is a section $f \in H^0(X_1(N), \omega^{\otimes k})$.

Remark. We will define "generalized" elliptic curves over S in later lectures, in order to get a proper $X_1(N)$ instead of $Y_1(N)$.

Katz gave these modular forms a new interpretation, and soon we will see that it leads to some concrete calculations.

Definition 1.2 (Katz modular forms)**.** A *test object* over a $\mathbb{Z}[\frac{1}{N}]$-algebra R, is a triple (E, i, ω), where

- (E, i) is an R-point of $\mathcal{M}_1(N)$,
- ω is a generator of the rank one free R-module $\omega_{E/R} = \pi_* \Omega_{E/R}$, where $\pi : E \to \operatorname{Spec} R$.

A *Katz modular form of weight k* is a rule which for any $\mathbb{Z}[\frac{1}{N}]$-algebra R, any test object (E, i, ω) over R, assigns a value $f(E, i, \omega) \in R$, such that,

(1) the assignment only depends on the isomorphism class of the test object (E, i, ω);

(2) f is compatible with base change $R \to R'$, i.e. for $\alpha : \mathrm{Spec}\ R' \to \mathrm{Spec}\ R$, we have

$$f(\alpha^* E, \alpha^* i, \alpha^* \omega) = \alpha^*(f(E, i, \omega)) \in R';$$

(3) f satisfies $f(E, i, a \cdot \omega) = a^{-k} f(E, i, \omega),\ \forall a \in R^\times$.

An immediate observation is that Katz's new definition makes no difference from the original one.

Theorem 1.3. *The space of modular forms is the same as the space of Katz modular forms.*

Proof. Indeed, given a usual modular form $f \in H^0(X_1(N), \omega^{\otimes k})$, we can obtain a Katz modular form f^{Katz} using the universal property of $X_1(N)$. More specifically, for any test object (E, i, ω), $\exists!$ morphism $\alpha : \mathrm{Spec}\ R \to X_1(N)$, such that $(E, i) = \alpha^*(\mathcal{E}_{\mathrm{univ}}, i_{\mathrm{univ}})$. Then $\alpha^*(f)$ is a section of $H^0(\mathrm{Spec}\ R, \omega_{E/R}^{\otimes k})$. So $\alpha^*(f) = s \cdot \omega^{\otimes k}$ for some $s \in R$. We set $f^{\mathrm{Katz}}(E, i, \omega) := s$.

It is easy to verify all of (1)-(3) holds by constuction. The converse also holds since one can use a Katz modular form f to construct compatible sections in $H^0(U, \omega^{\otimes k})$ for each affine subset U of $X_1(N)$. (The restriction of the universal bundle $(\mathcal{E}_{\mathrm{univ}}, i_{\mathrm{univ}})$ on U forms a test object.) $\square$

1.1 Application I: Hecke operators T_p over $\mathbb{Q}_p$

Suppose $p \nmid N$. In the language of usual modular forms, we can define the Hecke operators T_p as follow:

$$\begin{tikzcd} & X(\Gamma_1(N) \cap \Gamma_0(p)) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1(N) & & X_1(N) \end{tikzcd}$$

Since $X(\Gamma_1(N) \cap \Gamma_0(p))$ parametrizing the triple (E, i, C), where C is a subgroup of order p, there are 2 maps from $X(\Gamma_1(N) \cap \Gamma_0(p))$ to $X_1(N)$, namely

$$\pi_1 : (E, i, C) \mapsto (E, i); \quad \pi_2 : (E, i, C) \mapsto (E', i'),\ \text{where } E' = E/C,\ i' : \mu_{N,S} \xhookrightarrow{i} E \to E'.$$

To distinguish, we use the notation of the $\mathcal{E} = \mathcal{E}_{\text{univ}}, \omega$ on the $X_1(N)$ of π_1 side, and the notation of the $\mathcal{E}', \omega'$ on the $X_1(N)$ of π_2 side. Moreover, there exists a universal isogeny $\underbrace{\pi_1^*\mathcal{E} \longrightarrow \pi_2^*\mathcal{E}' \overset{\check{\pi}}{\dashrightarrow} \pi_1^*\mathcal{E}}_{\text{multiply by } p}$ So we get a pullback map $\check{\pi}^* : \pi_1^*\omega \to \pi_2^*\omega'$.

Definition 1.4. We define T_p' to be the map:

$$\begin{aligned} T_p' : H^0(X_1(N), \omega^{\otimes k}) &\to H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \pi_1^*\omega^{\otimes k}) \simeq H^0(X_1(N), \pi_{2*}\pi_1^*\omega^{\otimes k}) \\ &\xrightarrow{\check{\pi}^*} H^0(X_1(N), \pi_{2*}\pi_2^*\omega'^{\otimes k}) \xrightarrow{\mathrm{Tr}_{\pi_2}} H^0(X_1(N), \omega'^{\otimes k}) \end{aligned}$$

And we define the *Hecke operator* $T_p = \frac{1}{p} T_p'$.

It is equivalent to define more elegantly using the language of Katz modular forms as below.

Definition 1.5. Given a Katz modular form f, we define a Katz modular form $T_p(f)$ as follow:

For each test object (E, i, ω) over a $\mathbb{Z}[\frac{1}{Np}]$-algebra R, there are exactly $p+1$ subgroup scheme $C \subset E[p]$ of rank p over Spec R. (assuming Spec R is connected). Define

$$T_p(f)(E, i, \omega) := p^{k-1} \sum_{C \subset E[p]} f(E/C, i_C, \omega_C),$$

where we define $\check{\pi}$ to be the isogeny in $\underbrace{E \xrightarrow{\pi} E/C \overset{\check{\pi}}{\dashrightarrow} E}_{\text{multiply by } p}$, and let $\omega_C := \check{\pi}^*(\omega)$, $i_C : \mu_{N,S} \overset{i}{\hookrightarrow} E \overset{\pi}{\to} E/C$.

1.2 Application II: Tate curve and q-expansions

Recall the theory of elliptic curves over $\mathbb{C}$. Given a lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$, the quotient $\mathbb{C}/\Lambda_\tau$ is an elliptic curve E satisfying the equation $Y^2 = 4X^3 - \frac{E_4}{12}X + \frac{E_6}{216}$. This is given by the isomorphism

$$\begin{aligned} \mathbb{C}/\Lambda_\tau &\to E \\ z &\mapsto X = \wp(2\pi i z, 2\pi i \Lambda_\tau), Y = \wp'(2\pi i z, 2\pi i \Lambda_\tau) \end{aligned}$$

where the explicit formula is listed below:

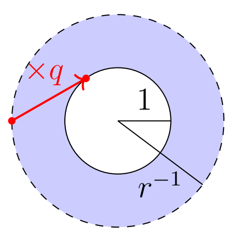
$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{l\in\Lambda-\{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2}\right)$$
$$\wp'(z,\Lambda) = \frac{d\wp(z,\Lambda)}{dz} = \sum_{l\in\Lambda-\{0\}} \frac{-2}{(z-l)^3}$$
$$E_4 = \frac{45}{\pi^2}\sum_{l\in\Lambda-\{0\}}\frac{1}{l^4} = 1+240\sum_{n\geq 1}\sigma_3(n)q^n \in \mathbb{Z}[[q]]$$
$$E_6 = \frac{945}{\pi^2}\sum_{l\in\Lambda-\{0\}}\frac{1}{l^6} = 1-504\sum_{n\geq 1}\sigma_5(n)q^n \in \mathbb{Z}[[q]]$$

In above formulas, $q = \exp(2\pi i\tau)$. We may further view the elliptic curve as

$$\begin{aligned} \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau) &\to \mathbb{C}^\times/q^{\mathbb{Z}} \\ z &\mapsto \exp(2\pi i z) \end{aligned}$$

Now as $E_4, E_6 \in \mathbb{Z}[[q]]$ (instead of $\mathbb{C}[[q]]$), so the *Tate curve* $\mathrm{Tate}_q := \mathbb{C}^\times/q^{\mathbb{Z}}$ can be actually defined over $\mathbb{Z}[\frac{1}{6}]((q))$ by the equation $Y^2 = 4X^3 - \frac{E_4}{12}X + \frac{E_6}{216}$.

Remark. (1) When $q=0$, $Y^2 = 4X^3 - \frac{1}{12}X + \frac{1}{216} = (X-\frac{1}{6})(X-\frac{1}{12})^2$ is singular.
(2) We may slightly change the coordinate by $X = x + \frac{1}{12}$, $Y = x + 2y$, to make the Tate curve defined over $\mathbb{Z}((q))$.
(3) There is a similar construction works for $\mathbb{C}_p$. Consider $q \in \mathbb{C}_p$, such that $|q|_p = r < 1$, then multiply by q folds the "annulus" $\{x \in \mathbb{C}_p | \ |x|_p \in [1, r^{-1}]\}$ into a proper rigid analytic space over $\mathbb{Q}_p$. By rigid GAGA, this defines an elliptic curve Tate_q over $\mathbb{Q}_p(q)$.



The Tate curve is equipped with a natural level structure $i_N : \mu_N \hookrightarrow \mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} = \mathrm{Tate}_q$. And there is a natural basis $\omega_{\mathrm{can}} := \frac{dX}{Y}$ of $\omega_{\mathrm{Tate}_q/\mathbb{Z}[\frac{1}{6}]((q))}$. Using the complex parametrizing above, we have

$$\omega_{\mathrm{can}} = \frac{dX}{Y} = 2\pi i dz = \frac{dz^\times}{z^\times}, \text{ where } z^\times := \exp(2\pi i z)$$

In terms of moduli problem, we get a pullback diagram by the universal property of $X_1(N)$.

$$\begin{tikzcd}
\mathrm{Tate}_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d] \\
\mathrm{Spec}\ Z((q)) \arrow[r] & X_1(N)
\end{tikzcd}$$

If f is a modular form of weight k, its evaluation on the object $(\mathrm{Tate}_q, i_N, \omega_{\mathrm{can}})$ is an element $f(\mathrm{Tate}_q, i_N, \omega_{\mathrm{can}}) \in \mathbb{Z}[\frac{1}{6N}]((q))$. This element is called the *q-expansion of f*.

Fact (The q-expansion principle)**.** *If the q-expansion of a modular form f over $X_1(N)$ is identically zero, then $f = 0$.*

With this fact, to show that the two different definitions of the Hecke operators given in section 1.1 are the same, we only need to check whether the q-expansion of Katz's version of T_p agree with the usual one.

Some Calculations. Now we compute

$$T_p(f)(\mathrm{Tate}_q, i_N, \omega_{\mathrm{can}}) = p^{k-1} \sum_{C \subset \mathrm{Tate}_q[p]} f(\mathrm{Tate}_q/C, \pi^* i_N, \check{\pi}^* \omega_{\mathrm{can}}).$$

By abusing the notation, the q-expansion of f, $f(\mathrm{Tate}_q, i_N, \omega_{\mathrm{can}}) \in \mathbb{Z}[\frac{1}{6N}]((q))$ is also denoted by $f = f(q)$.

- Case 1. $C = \mu_p$. In this case, we have
  - $\mathrm{Tate}_q/\mu_p \simeq \mathbb{C}^\times/(q^{\mathbb{Z}}\mu_p) \xrightarrow[x \mapsto x^q]{\simeq} \mathbb{C}^\times/q^{p\mathbb{Z}} = \mathrm{Tate}_{q^p}$.
  - $\check{\pi} : \mathbb{C}/q^{p\mathbb{Z}} \to \mathbb{C}^\times/q^{\mathbb{Z}}$ is the natural quotient, so $\check{\pi}^* \frac{dz^\times}{z^\times} = \frac{dz^\times}{z^\times}$.
  - $\pi^* i_N : \mu_N \to \mathbb{C}^\times/q^{\mathbb{Z}} \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{p\mathbb{Z}}$, this is $\langle p \rangle i_N := (x \mapsto x^p) \circ i_N$, where $\langle p \rangle$ is called the diamond operator, it also acts on modular forms by $\langle p \rangle f(E, i, \omega) := f(E, \langle p \rangle i, \omega)$. [1]

[1] $\langle p \rangle$ also has an Adelic definition: For a modular form

$$f : \mathrm{GL}_2(\mathbb{Q}) \Big\backslash \mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f) \Big/ \widehat{\Gamma_1(N)} \to \mathbb{C}$$

we define $\langle p \rangle (f)(z) := f\left(z \cdot \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)$. One could check that it gives the modular interpretation above.

Hence in this case, we get

$$f(\mathrm{Tate}_q/C, \pi^* i_N, \check{\pi}^* \omega_{\mathrm{can}}) = f(\mathrm{Tate}_{q^p}, \langle p \rangle i_N, \omega_{\mathrm{can}/\mathrm{Tate}_{q^p}}) = \langle p \rangle f(q^p)$$

- Case 2. $C = \langle \zeta_p^i q^{1/p} \rangle$, for $i = 0, 1, ..., p-1$.
  - $\mathrm{Tate}_q/C \simeq \mathbb{C}^\times/(\zeta_p^i q^{1/p})^{\mathbb{Z}} = \mathrm{Tate}_{\zeta_p^i q^{1/p}}$.
  - $\check{\pi} : \mathbb{C}^\times/(\zeta_p^i q^{1/p})^{\mathbb{Z}} \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{\mathbb{Z}}$ is the map raising to the p-th power, so $\check{\pi}^* \frac{dz^\times}{z^\times} = p \cdot \frac{dz^\times}{z^\times}$.
  - $\pi^* i_N : \mu_N \to \mathbb{C}^\times/q^{\mathbb{Z}} \to \mathbb{C}^\times/(\zeta_p^i q^{1/p})^{\mathbb{Z}}$, is the natural one on $\mathrm{Tate}_{\zeta_p^i q^{1/p}}$.

  Hence in this case, we get

$$f(\mathrm{Tate}_q/C, \pi^* i_N, \check{\pi}^* \omega_{\mathrm{can}}) = f(\mathrm{Tate}_{\zeta_p^i q^{1/p}}, i_{N/\mathrm{Tate}_{\zeta_p^i q^{1/p}}}, p \cdot \omega_{\mathrm{can}/\mathrm{Tate}_{\zeta_p^i q^{1/p}}}) = p^{-k} f(\zeta_p^i q^{1/p}).$$

To summarize,

$$T_p(f) = p^{k-1} \langle p \rangle f(q^p) + p^{-1} \sum_{i=0}^{p-1} f(\zeta_p^i q^{1/p}).$$

This is exactly the usual formula on q-expansions.

2 Kodaira-Spencer isomorphism

Now let k be a field of characteristic 0. Recall that if X is a proper smooth variety over k, then we have its de Rham cohomology

$$H^\bullet_{\mathrm{dR}}(X/k) := \mathbb{H}^\bullet(X, \mathcal{O}_X \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots)$$

Each $H^n_{\mathrm{dR}}(X/k)$ carries a natural decreasing filtration

$$\underbrace{\underbrace{\underbrace{\underbrace{H^0(X, \Omega^n_{X/k})}_{\mathrm{Fil}^n} - H^1(X, \Omega^{n-1}_{X/k})}_{\mathrm{Fil}^{n-1}} - H^2(X, \Omega^{n-2}_{X/k})}_{\mathrm{Fil}^{n-2}} - \cdots - H^n(X, \mathcal{O}_X)}_{\substack{\ddots \\ \mathrm{Fil}^0}}$$

Remark. Although as k-vector spaces, $H^n_{\mathrm{dR}}(X/k) \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/k})$, the decompostion is *not* canonical (or functorial). Only the Hodge filtration is canonical.

Recall from last lecture that in general, if $X \xrightarrow{\pi} S$ is a proper smooth family, we can define $\mathcal{H}^\bullet_{\mathrm{dR}}(X/S) := R\pi_*(\Omega^\bullet_{X/S})$. At least in characteristic 0, each $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ is a vector bundle over S, carrying a similar filtration

$$\underbrace{\underbrace{\underbrace{\underbrace{\pi_*\Omega^n_{X/k}}_{\mathrm{Fil}^n} - R^1\pi_*\Omega^{n-1}_{X/k}}_{\mathrm{Fil}^{n-1}} - R^2\pi_*\Omega^{n-2}_{X/k}}_{\mathrm{Fil}^{n-2}} - \cdots - R^n\pi_*\mathcal{O}_X}_{\substack{\ddots \\ \mathrm{Fil}^0}}$$

And it admits an additional feature: *the Gauss-Manin connection.*

$$\nabla : \mathcal{H}^n_{\mathrm{dR}}(X/S) \to \mathcal{H}^n_{\mathrm{dR}}(X/S) \otimes \Omega_{S/k}$$

- it is integrable: $\nabla^2 = 0$,
- it satisfies Griffiths transversality: $\nabla(\mathrm{Fil}^i\mathcal{H}^n_{\mathrm{dR}}(X/S)) \subseteq \mathrm{Fil}^{i-1}\mathcal{H}^n_{\mathrm{dR}}(X/S) \otimes \Omega^1_{S/k}$.

Now we are interested in the case of modular curves, let $X = \mathcal{E},\ S = Y_1(N)$. In this case, we have

$$\begin{tikzcd}
\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/Y_1(N)) \arrow[r] \arrow[d, equal] & \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d, phantom, "\shortparallel"] \\
\omega_{\mathcal{E}/Y_1(N)} \arrow[r, "\text{Griffiths}", "\text{transversality}"'] \arrow[d, dash] & \left(\begin{array}{c} \omega_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k} \\ | \\ \omega^{-1}_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k} \end{array} \right. \\
\omega^{-1}_{\mathcal{E}/Y_1(N)} &
\end{tikzcd}$$

Now by the above diagram, we get a map

$$\begin{tikzcd}
\omega_{\mathcal{E}/Y_1(N)} \arrow[r] \arrow[dr] & \mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d, two heads] \\
& \mathrm{gr}^0\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k}
\end{tikzcd}$$

i.e. $\omega \to \omega^{-1} \otimes \Omega^1_{Y_1(N)/k}$. Note that this map is a map of coherent sheaves, *not* a differential map. Tensoring with ω we get the Kodaira-Spencer map

$$\mathrm{KS} : \omega^{\otimes 2} \to \Omega^1_{Y_1(N)/k}.$$

Theorem 2.1 (Kodaira-Spencer isomorphism)**.** *The KS map defined above induces an isomorphism*

$$KS : \omega^{\otimes 2} \to \Omega^1_{X_1(N)/k}(\log C) = \Omega^1_{X_1(N)/k}(C)$$

where C represents the cusps, and the $\log C$ *means with log pole at cusps.*

Moreover, this isomorphism even extends to $\mathbb{Z}[\frac{1}{N}]$.

3 De Rham cohomology for local systems

Now $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{E}/Y_1(N))$ is locally free of rank 2, carrying an integrable Gauss-Manin connection. For $k \geq 2$, we get the symmetric power $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}$, with an integrable connection.

Remark. This naturally extends to the cusps, which we will discuss in a later lecture. For present, we just ignore the cusp problem and pretend that $Y_1(N)$ is proper and write $X_1(N)$ instead.

The goal of this section is to understand

$$H^*_{\mathrm{dR}}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) := \mathbb{H}^*(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}(\log C))$$

Over $\mathbb{C}$, by de Rham-Betti comparison, and the comparison to étale cohomology (by choosing an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_l$), we have

$$\begin{aligned} H^*_{\mathrm{dR}}(X_1(N)_{\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) &\simeq H^*(X_1(N)(\mathbb{C}), \mathrm{Sym}^{k-2}(R^1\pi_*\underline{\mathbb{C}})) \\ &\simeq H^*_{\text{ét}}(X_1(N), \mathrm{Sym}^{k-2}(R^1\pi_*\underline{\bar{\mathbb{Q}}_l})) \end{aligned}$$

Warning. This isomorphism holds for $Y_1(N)$, but may have problems for $X_1(N)$.

Recall the Hodge filtration $\omega - \omega^{-1}$ on $\mathcal{H}^1_{\mathrm{dR}}$, it induces a natural filtration on $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}$.

$$\begin{array}{ccccccccccc} & \mathrm{Fil}^{k-2} & & \mathrm{Fil}^{k-3} & & \mathrm{Fil}^{k-4} & & & & & \\ \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}: & \omega^{k-2} & - & \omega^{k-4} & - & \omega^{k-6} & - & \cdots & - & \omega^{2-k} \\ & \| & & \| & & \| & & & & \| \\ & \omega^{k-2} & & \omega^{k-3} \otimes \omega^{-1} & & \omega^{k-4} \otimes \omega^{-2} & & \cdots & & \omega^{-(k-2)} \end{array}$$

By Kodaira-Spencer isomorphism, $\Omega^1_{X_1(N)}(\log C) \simeq \omega^2$, so we have a filtration of the same length on $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}(\log C)$.

$$\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}(\log C): \quad \omega^k - \omega^{k-2} - \omega^{k-4} - \cdots - \omega^{4-k}$$

By Griffiths transversality, we have

$$\nabla(\mathrm{Fil}^i(\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}})) \subseteq \mathrm{Fil}^{i-1}(\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) \otimes \Omega^1_{X_1(N)}(\log C)$$

$$
\begin{tikzcd}
\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} : \arrow[d, "\nabla"] & \omega^{k-2} \arrow[dr, "\wr"] \arrow[r, dash] & \omega^{k-4} \arrow[dr, "\wr"] \arrow[r, dash] & \omega^{k-6} \arrow[dr, "\wr"] \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & \omega^{2-k} \\
\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}(\log C) : & \omega^{k} \arrow[r, dash] & \omega^{k-2} \arrow[r, dash] & \omega^{k-4} \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & \omega^{4-k}
\end{tikzcd}
$$

As a corollary of Kodaira-Spencer isomorphism, the corresponding graded map

$$\mathrm{gr}^i(\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) \to \mathrm{gr}^{i-1}(\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) \otimes \Omega^1_{X_1(N)}(\log C)$$

is an isomorphism.

Thus we can get a quasi-isomorphism between the following complexes

$$[\omega^{2-k} \xrightarrow{\theta} \omega^k] \xrightarrow{\text{q.isom.}} [\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}(\log C)]$$

where the θ map is given by: given a section s of ω^{2-k}, lift it to the *unique* element $\tilde{s} \in \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}$, such that $\nabla(\tilde{s}) \in \omega^k$. We define $\theta(s) := \nabla(\tilde{s})$.

Remark. In the exercise we will see that on the level of q-expansions,

$$\theta(f) = \frac{(-1)^{k-2}}{(k-2)!}\left(q\frac{d}{dq}\right)^{k-1}(f).$$

From the quasi-isomorphism, we get the following result.

Theorem 3.1. $H^*_{\mathrm{dR}}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) \simeq \mathbb{H}^*(X_1(N), \omega^{2-k} \xrightarrow{\theta} \omega^k)$

Now assume that $k \geq 3$ for simplicity. By spectral sequence of the hypercohomology, the E_1-page is

$$H^1(X_1(N), \omega^{2-k}) \xrightarrow{\theta} H^1(X_1(N), \omega^k)$$

$$H^0(X_1(N), \omega^{2-k}) \xrightarrow{\theta} H^0(X_1(N), \omega^k)$$

But by Serre duality and Kodaira-Spencer isomorphism, we have

$$H^1(X_1(N), \omega^k) \simeq H^0(X_1(N), \omega^{2-k}(-C))^\vee.$$

Since there is no modular forms of weight less than 0, both terms $H^1(X_1(N), \omega^k)$ and $H^0(X_1(N), \omega^{2-k})$ are zero. This implies the convergence of the E_1-page. Moreover, in the notation of spaces of modular forms, $H^0(X_1(N), \omega^k) = \mathcal{M}_k(\Gamma_1(N))$, $H^1(X_1(N), \omega^{2-k}) \simeq H^0(X_1(N), \omega^k(-C))^\vee = \mathcal{S}_k(\Gamma_1(N))^\vee$. Thus we get an exact sequence

$$0 \to \mathcal{M}_k(\Gamma_1(N)) \to H^1_{\mathrm{dR}}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}) \to \mathcal{S}_k(\Gamma_1(N))^\vee \to 0$$

This is the Eichler-Shimura isomorphism.

Remark. A generaliztion of this is called *the dual-BGG complex* of Faltings. He interpreted the above phenomenon in terms of Bernstein-Gelfand-Gelfand complex from representations of Lie groups.