Geometric properties of modular forms

§1 Katz' interpretation of modular forms

We will explain this using the moduli space $X_1(N)$ as an example, $N \geq 4$.

Recall $X_1(N)$ represents the functor

$$\mathcal{M}_1(N): \quad \mathrm{Sch}/\mathbb{Z}[\tfrac{1}{N}] \longrightarrow \mathcal{S}ets$$

$$S \longmapsto \mathcal{M}_1(N)(S) = \left\{ \begin{array}{l} \text{isom. classes of } (E, i) \\ E \text{ a "generalized" elliptic curve over } S \\ i : \mu_{N/S} \hookrightarrow E[N] \end{array} \right\}$$

will explain this in later sections, purpose is to define a proper $X_1(N)$

A weight k modular form is a section $f \in H^0(X_1(N), \omega^{\otimes k})$.

Katz's new definition: A test object over a $\mathbb{Z}[\frac{1}{N}]$-algebra R is a triple (E, i, ω), where

- (E, i) is an R-point of $\mathcal{M}_1(N)$
- ω is a generator of the free rank one R-module $\omega_{E/R}$.

A Katz modular form of weight k is a rule to associate to

$$\left.\begin{array}{l} \bullet \text{ every } \mathbb{Z}[\frac{1}{N}]\text{-algebra } R \text{. and} \\ \bullet \text{ every test object } (E, i, \omega) \end{array}\right\} \text{ an element } f(E, i, \omega) \in R$$

s.t. (1) This assignment depends only on isom. class of (E, i, ω)

(2) is compatible with base change in R,

i.e. for $\mathrm{Spec}\, R' \xrightarrow{\alpha} \mathrm{Spec}\, R$, $f(\alpha^* E, \alpha^* i, \alpha^* \omega) = \alpha^*(f(E, i, \omega)) \in R'$

(3) satisfies $f(E, i, a \cdot \omega) = a^{-k} f(E, i, \omega)$ for $a \in R^\times$

Theorem. The space of modular forms is the same as the space of Katz modular forms

Indeed, given a usual modular form $f \in H^0(X_1(N), \omega^{\otimes k})$, we obtain a Katz modular form f^{Katz}:

for every test object (E, i, ω) over R,

$\exists !$ morphism $\alpha : \mathrm{Spec}\, R \longrightarrow X_1(N)$ s.t. $(E, i) = \alpha^*(\mathcal{E}_{\mathrm{univ}}, i_{\mathrm{univ}})$

Then $\alpha^*(f)$ is a section of $H^0(\mathrm{Spec}\, R, \omega_{E/R}^{\otimes k})$

so $\alpha^*(f) = s \cdot \omega^{\otimes k}$ for some $s \in R$ $\leadsto$ set $f^{\text{Katz}}(E,i,\omega) := s$.

Properties (1) (2) (3) are easy to see. & the converse is also immediate.

- <u>Application I.</u> Describe Hecke operators T_p-action on Katz modular form f over $\mathbb{Q}_p$ $\quad p \nmid N$

Given a Katz modular form f, we define a new Katz modular form $T_p(f)$ as follows:

For each test object (E, i, ω) over a $\mathbb{Z}[\frac{1}{Np}]$-algebra R (assuming Spec R is connected)

there are exactly $p+1$ subgroup schemes $C \subset E[p]$ of rank p over Spec R.

Define $$T_p(f)(E,i,\omega) := \overset{\text{normalizing factor}}{p^{k-1}} \sum_{C \subset E[p]} f(E/C, i_C, \omega_C)$$

where ω_C is given as follows: $\begin{tikzcd} E \arrow[r, "\pi"] \arrow[rr, bend left, "\text{mult by } p"] & E/C \arrow[r, dashed, "\check{\pi}"'] & E \end{tikzcd}$, $i_C : \mu_{N,S} \xhookrightarrow{i} E \xrightarrow{\pi} E/C$

define $\omega_C := \check{\pi}^*(\omega)$

We can alternatively define this as follows:

$$\begin{tikzcd} & X(\Gamma_1(N) \cap \Gamma_0(p)) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1(N) & & X_1(N) \end{tikzcd}$$

$X(\Gamma_1(N) \cap \Gamma_0(p))$ — parametrizing (E, i, C) with C a subgroup of order p

equivalent to $(E \underset{\text{isog.}}{\xrightarrow{\pi}} E', i)$

s.t. $\text{Ker}(E \xrightarrow{\pi} E')$ has degree p.

(E, i) on $X_1(N)$

$\pi_2(E \xrightarrow{\pi} E', i) = (E', i')$ where $i' : \mu_{N/S} \xhookrightarrow{i} E \xrightarrow{\pi} E'$

Note that there's a universal isogeny $\begin{tikzcd} \pi_1^*\mathcal{E} \arrow[r] \arrow[rr, bend right, "\text{mult.} p"'] & \pi_2^*\mathcal{E}' \arrow[r, dashed, "\check{\pi}"] & \pi_1^*\mathcal{E} \end{tikzcd}$

Pulling back along $\check{\pi}$, get $\check{\pi}^* : \pi_1^*\omega \longrightarrow \pi_2^*\omega'$

We define T_p-operator as:

$$T_p' : \underbrace{H^0(X_1(N), \omega^{\otimes k})}_{\text{on the } \pi_1 \text{ side}} \longrightarrow H^0(X(\Gamma_1(N) \cap \Gamma_0(p)), \pi_1^*\omega^{\otimes k}) \cong H^0(\underbrace{X_1(N)}_{\text{on the } \pi_2 \text{ side}}, \pi_{2*}\pi_1^*\omega^{\otimes k})$$

$$\xrightarrow{\check{\pi}^*} H^0(X_1(N), \pi_{2*}\pi_2^*\omega'^{\otimes k}) \xrightarrow{Tr_{\pi_2}} H^0(X_1(N), \omega'^{\otimes k})$$

Define $T_p := \frac{1}{p} \cdot T_p'$. The normalization factor $\frac{1}{p}$ is very important!

<u>Remark</u>: Recently, Fakhruddin & Pilloni defined these Hecke operators integrally (over $\mathbb{Z}_p$) for

general Shimura varieties of PEL type and all weights.

Application II. Tate curve & q-expansion explained. (following Katz)

Over $\mathbb{C}$: Given a lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, the quotient $\mathbb{C}/\Lambda_\tau$ is the elliptic curve $Y^2 = 4X^3 - \frac{E_4}{12}X + \frac{E_6}{216}$

$$\mathbb{C}/\Lambda_\tau \ni z \longmapsto x = \wp(2\pi i z, 2\pi i \Lambda_\tau),\ y = \wp'(2\pi i z, 2\pi i \Lambda_\tau)$$

where
$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\ell \in \Lambda - \{0\}} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right)$$
$$\wp'(z,\Lambda) = \frac{d\wp(z;\Lambda)}{dz} = \sum_{\ell \in \Lambda} \frac{-2}{(z-\ell)^3}$$
$$E_4 = \frac{45}{\pi^4} \sum_{\ell \in \Lambda_\tau - \{0\}} \frac{1}{\ell^4} = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \in \mathbb{Z}[\![q]\!]$$
$$E_6 = \frac{945}{2\pi^6} \sum_{\ell \in \Lambda_\tau - \{0\}} \frac{1}{\ell^6} = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n \in \mathbb{Z}[\![q]\!]$$

When $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, we can view elliptic curve as

$$\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \xrightarrow[\cong]{\exp(2\pi i \cdot)} \mathbb{C}^\times / q^{\mathbb{Z}} \quad \text{for } q = e^{2\pi i \tau}$$

$(x - \frac{1}{6})(x - \frac{1}{12})^2$

$\shortparallel$

when $q = 0$, $Y^2 = 4X^3 - \frac{1}{12}X + \frac{1}{216}$ is singular.

As $E_4, E_6 \in \mathbb{Z}[\![q]\!]$, so the Tate curve $\mathrm{Tate}_q := \mathbb{C}^\times / q^{\mathbb{Z}}$ is def'd over $\mathbb{Z}[\frac{1}{6}]((q))$

Rmk: can change coordinates to be def'd over $\mathbb{Z}((q))$

Remark: This analytic construction also works over $\mathbb{C}_p$ $\quad |q|_p = r < 1$.

mult. by q folds the annulus with radius in $[1, r^{-1}]$ into a rigid analytic space over $\mathbb{Q}_p$

By rigid GAGA, this defines an elliptic curve Tate_q over $\mathbb{Q}_p[\![q]\!]$

This Tate curve is equipped with a natural level structure

$$i_N : \mu_N \hookrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}^\times / q^{\mathbb{Z}} = \mathrm{Tate}_q$$

There's a natural basis $\frac{dX}{Y}$ of $\omega_{\mathrm{Tate}_q / \mathbb{Z}[\frac{1}{6}][\![q]\!]}$

invariant differential on $\mathbb{C}^\times$.

Using the parametrization above $\frac{dX}{Y} = 2\pi i\, dz = \boxed{\frac{dz^\times}{z^\times}} =: \omega_{can}$ $\quad$ where $z^\times := \exp(2\pi i z)$

In terms of the moduli problem, we get a morphism (& a Cartisian pullback diagram)

$$\begin{tikzcd}
E_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d, dash] \\
\mathrm{Spec}\, \mathbb{Z}((q)) \arrow[r] & X_1(N)
\end{tikzcd}$$

If f is a modular form of wt k, its evaluation on the object $(\mathrm{Tate}_q, i_N, \omega_{can})$

is $\quad f(\mathrm{Tate}_q, i_N, \omega_{can}) \in \mathbb{Z}[\frac{1}{6N}][[q]]$

This is the q-expansion of f.

<u>We compute</u>: $T_p(f)(\mathrm{Tate}_q, i_N, \omega_{can}) = p^{k-1} \sum_{C \subset E_q[p]} f\left(\mathrm{Tate}_q/C,\ i_N',\ \check{\pi}^* \omega_{can}\right)$

* <u>Case 1</u>. $C = \mu_p$, $\mathrm{Tate}_q/\mu_p \cong \mathbb{C}^\times/q^{\mathbb{Z}}\cdot\mu_p \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{p\mathbb{Z}}$

$\check{\pi} : \mathbb{C}^\times/q^{p\mathbb{Z}} \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ is the natural quotient

so $\check{\pi}^* \frac{dz^\times}{z^\times} = \frac{dz^\times}{z^\times}$.

$i_N' : \mu_N \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}} \xrightarrow{x \mapsto x^p} \mathbb{C}^\times/q^{p\mathbb{Z}}$ is the $\overset{\text{Diamond operator}}{\swarrow}\langle p \rangle \cdot i_N$

* <u>Case 2</u> $C = \langle \zeta_p^i q^{\frac{1}{p}} \rangle$ for $i = 0, 1, \cdots, p-1$

$$\mathrm{Tate}_q/C \cong \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}} = \mathrm{Tate}_{\zeta_p^i q^{\frac{1}{p}}}$$

$\check{\pi} : \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}} \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ is raising to p^{th} power

$\Rightarrow \check{\pi}^* \frac{dz^\times}{z^\times} = p \cdot \frac{dz^\times}{z^\times}$ so $\check{\pi}^* \omega_{can} = p \cdot \omega_{can}$

$i_N' : \mu_N \to \mathbb{C}^\times/q^{\mathbb{Z}} \longrightarrow \mathbb{C}^\times/(\zeta_p^i q^{\frac{1}{p}})^{\mathbb{Z}}$ is the natural one.

So we have $T_p(f) = p^{k-1} \langle p \rangle f(q^p) + p^{k-1} \cdot \sum_{i=0}^{p-1} \underbrace{p^{-k}}_{\uparrow} \cdot f(\zeta_p^i q^{\frac{1}{p}})$

from $\check{\pi}^* \omega_{can} = p \cdot \omega_{can}$

This is exactly the usual formula on q-expansions.

§2 Kodaira-Spencer isomorphism

Let k be a field of char 0.

- If X is a proper smooth variety over k, then we have its de Rham cohomology

$$H^\bullet_{dR}(X/k) := \mathbb{H}^\bullet\left(X, \mathcal{O}_X \to \Omega^1_{X/k} \to \underset{\overset{\parallel}{\Lambda^2 \Omega^1_{X/k}}}{\Omega^2_{X/k}} \to \cdots\right)$$

Each $H^n_{dR}(X/k)$ carries a natural decreasing filtration

$$H^0(X, \Omega^n_{X/k}) \;—\; H^1(X, \Omega^{n-1}_{X/k}) \;—\; H^2(X, \Omega^{n-2}_{X/k}) \;—\; \cdots$$

$\underbrace{\qquad}_{\mathrm{Fil}^n}$

$\underbrace{\qquad\qquad}_{\mathrm{Fil}^{n-1}}$

$\underbrace{\qquad\qquad\qquad}_{\mathrm{Fil}^{n-2}}$

Upshot : As k-vector spaces, $H^n_{dR}(X/k) \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/k})$ but not canonically.

Only the Hodge filtration is canonical.

In general, if $\begin{matrix} X \\ \downarrow \pi \\ S/k \end{matrix}$ is a proper smooth family, can define $R\pi_*(\Omega^\bullet_{X/S}) =: \mathcal{H}^\bullet_{dR}(X/S)$

At least in char 0, each $\mathcal{H}^n_{dR}(X/S)$ is a vector bundle over S, carrying a filtration $\underbrace{\underbrace{\mathcal{H}^0(X,\Omega^n_{X/S})}_{\mathrm{Fil}^n} — \mathcal{H}^1(X,\Omega^{n-1}_{X/S})}_{\mathrm{Fil}^{n-1}} — \mathcal{H}^2(X,\Omega^{n-2}_{X/S})$ . . . .

There's an additional feature : Gauss-Manin connection

$$\nabla : \mathcal{H}^n_{dR}(X/S) \longrightarrow \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/k}$$

* it is integrable : $\nabla^2 = 0$ $\quad$ $$\begin{tikzcd} \mathcal{H}^n_{dR}(X/S) \arrow[r, "\nabla"] \arrow[rr, bend right, "0"'] & \mathcal{H}^n_{dR}(X/S) \otimes \Omega^1_{S/k} \arrow[r, "\nabla"] & \mathcal{H}^n_{dR}(X/S) \otimes \Omega^2_{S/k} \end{tikzcd}$$

So can talk about the de Rham complex $(\mathcal{H}^n_{dR}(X/S) \otimes \Omega^\bullet_{S/k}, \nabla)$

but we don't discuss this here.

* it satisfies Griffith transversality :

$$\nabla(\mathrm{Fil}^i \mathcal{H}^n_{dR}(X/S)) \subseteq \mathrm{Fil}^{i-1} \mathcal{H}^n_{dR} \otimes \Omega^1_{S/k}$$

For modular curves $\begin{matrix} X & & \mathcal{E} \\ \downarrow & = & \downarrow \\ S & & Y_1(N) \end{matrix}$

$$\begin{tikzcd} \nabla : \mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N)) \arrow[r] \arrow[d, equal] & \mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d, equal] \\ \omega_{\mathcal{E}/Y_1(N)} \arrow[d, dash] \arrow[r] \arrow[r, bend left, "\text{Griffith transversality}"] & \omega_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k} \arrow[d, dash] \\ \omega^{-1}_{\mathcal{E}/Y_1(N)} & \omega^{-1}_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k} \end{tikzcd}$$

Moreover, $$\begin{tikzcd} \omega_{\mathcal{E}/Y_1(N)} \arrow[r] \arrow[dr] & \mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d] \\ & gr^0 \mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \end{tikzcd}$$

This map is
a map of coherent sheaves
no differential maps

$$\Rightarrow \omega \longrightarrow \omega^{-1} \otimes \Omega^1_{Y_1(N)/k}$$

$$\Rightarrow KS: \omega^{\otimes 2} \longrightarrow \Omega^1_{Y_1(N)/k}$$

Theorem (Kodaira-Spencer isomorphism) KS induces an isomorphism

$$KS: \omega^{\otimes 2} \xrightarrow{\cong} \Omega^1_{X_1(N)/k}(\underbrace{\log C}_{\text{with log pole at cusps}}) = \Omega^1_{X_1(N)/k}(C)$$

Moreover, this isomorphism even extends to $\mathbb{Z}[\frac{1}{N}]$.

§3 de Rham cohomology of local systems

- $$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ Y_1(N) \end{tikzcd}$$ $\mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N))$ is locally free of rank 2 carrying an integrable connection.

For $k \geq 2$, get symmetric power $\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}$, with integrable connection.

(This extends naturally to the cusp, which we will discuss later)

We ignore the issue at the cusp & pretend that $Y_1(N)$ is proper & write $X_1(N)$ instead.

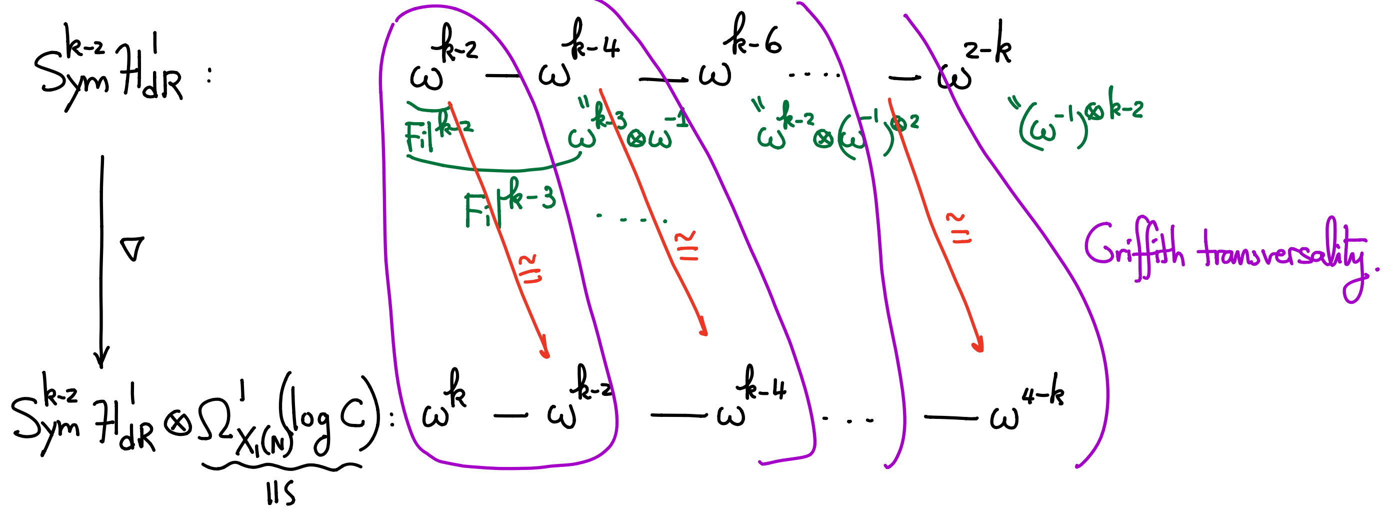
Goal: Understand $H^*(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \xrightarrow{\nabla} \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \otimes \Omega^1_{X_1(N)}(\log C)) =: H^*_{dR}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR})$

Over $\mathbb{C}$, by deRham-Betti comparison,

$$H^*_{dR}(X_1(N)_{\mathbb{C}}, \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}) \cong H^*(X_1(N)(\mathbb{C}), \mathrm{Sym}^{k-2}(\underbrace{R^1\pi_*\underline{\mathbb{C}}}_{\uparrow \text{rank 2 local system}}))$$

$$\overset{\text{choosing } \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell}{\cong} H^*_{et}(X_1(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}(R^1\pi_*\underline{\overline{\mathbb{Q}}_\ell}))$$

* The Hodge filtration $\omega - \omega^{-1}$ on $\mathcal{H}^1_{dR}$ induces a natural filtration



ω^2

The corresponding graded map $gr^i(\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}) \longrightarrow gr^{i-1}(\mathrm{Sym}^{k-2}\mathcal{H}^i_{dR}) \otimes \Omega^1_{X_1(N)}(\log C)$

is an isomorphism.

$\Rightarrow$ So the whole complex $\left[\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \xrightarrow{\nabla} \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \otimes \Omega^1_{X_1(N)}(\log C)\right] \overset{q.\mathrm{isom.}}{\xleftarrow{\sim}} \left[\omega^{2-k} \xrightarrow{\theta} \omega^k\right]$

where θ-map is given by sending a section s of ω^{2-k}

$\rightarrow$ lifts to $\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}$ $\rightarrow$ modify it into a <u>unique</u> lift $\tilde{s}$ such that $\nabla(\tilde{s}) \subseteq \omega^k$.

Define $\theta(s) := \nabla(\tilde{s})$.

Fact: On the level of q-expansions, $\theta(f) = \frac{(-1)^{k-2}}{(k-2)!}\left(q\frac{d}{dq}\right)^{k-1}(f)$.

Theorem: $H^*_{dR}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}) \cong \mathbb{H}^*(X_1(N), \omega^{2-k} \xrightarrow{\theta} \omega^k)$

Assume $k \geq 3$ for simplicity,

By spectral sequence,

$$E_1\text{-page} \quad \left| \begin{tikzcd} H^1(X_1(N), \omega^{2-k}) \arrow[r, "\theta"] & H^1(X_1(N), \omega^k)^{=0} \\ H^0(X_1(N), \omega^{2-k}) \arrow[r, "\theta"] & H^0(X_1(N), \omega^k) \end{tikzcd} \right. \Rightarrow H^*_{dR}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR})$$

$$\begin{tikzcd} \Rightarrow \quad 0 \arrow[r] & H^0(X_1(N), \omega^k) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^1_{dR}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}) \arrow[r] & H^1(X_1(N), \omega^{2-k}) \arrow[r] \arrow[d, phantom, "\wr\shortparallel"] & 0 \\ & M_k(\Gamma_1(N)) & & H^0(X_1(N), \omega^k(-C))^\vee \arrow[d, phantom, "\shortparallel"] & \\ & & & S_k(\Gamma_1(N))^\vee & \end{tikzcd}$$

This is the Eichler-Shimura isomorphism.

Generalizations of this is called <u>dual-BGG complex</u> of Faltings

He interpreted the above phenomenon in terms of Bernstein-Gelfand-Gelfand complex from rep'ns of Lie groups.