Overconvergent modular forms

§1. Hasse invariants for modular forms

Let S be an $\mathbb{F}_p$-scheme.

There's a Frobenius endomorphism $S \xrightarrow{\mathrm{Fr}_S} S$

If A is an abelian S-scheme, we have a relative Frobenius

$$\begin{tikzcd} A \arrow[dr, dashed, "\mathrm{Fr}_{A/S}"] \arrow[drr, bend left, "\mathrm{Fr}_A"] \arrow[ddr, "\pi"'] & & \\ & A^{(p)} \arrow[r, "\mathrm{Fr}_S"] \arrow[d, "\pi^{(p)}"] & A \arrow[d, "\pi"] \\ & S \arrow[r, "\mathrm{Fr}_S"] & S \end{tikzcd}$$

$$\begin{tikzcd} \sum s_n^p x^{np} & & \\ & \sum s_n^p x^n \arrow[ul, dashed] \arrow[l, leftarrow] & \sum s_n x^n \arrow[l] \arrow[ull, bend right] \end{tikzcd}$$

"Frobenius on the fiber"

Fact: $\mathrm{Ker}(\mathrm{Fr}_{A/S}) \subseteq A[p]$, so we have a factorization

$$\begin{tikzcd} A \arrow[r, "\mathrm{Fr}_{A/S}"] \arrow[rr, bend right, "\times p"'] & A^{(p)} \arrow[r, "V"] & A \end{tikzcd} \quad /S \qquad \text{all } S\text{-morphisms}$$

called Verschiebung.

Then $H^1_{dR}(A/S) \xrightarrow{V^*} H^1_{dR}(A^{(p)}/S) \cong H^1_{dR}(A/S) \otimes_{\mathcal{O}_S, \mathrm{Fr}_S} \mathcal{O}_S$

General Fact: $\mathrm{Im}\, V^*$ is precisely $\omega_{A^{(p)}/S} \cong \omega_{A/S} \otimes_{\mathcal{O}_S, \mathrm{Fr}_S} \mathcal{O}_S$

Example: modular curve $\quad p \nmid N$

$$\begin{tikzcd} \mathcal{E} \arrow[d] & E \arrow[l] \arrow[d] \\ \mathcal{Y}_1(N) \arrow[d] & Y_1(N) \arrow[l] \arrow[d] \\ \mathrm{Spec}\, \mathbb{Z}_{(p)} & \mathrm{Spec}\, \mathbb{F}_p \arrow[l] \end{tikzcd}$$

$\leftarrow$ special fiber.

to be discussed next week.

$\mathcal{X}_1(N)$ & $X_1(N)$ are "compactifications"

$M_k(\Gamma_1(N)) := H^0(\mathcal{X}_1(N), \omega^{\otimes k})$

$M_k(\Gamma_1(N), \mathbb{F}_p) := H^0(X_1(N), \omega_{\mathbb{F}_p}^{\otimes k})$

* Applying the above construction to $E \to Y_1(N)$

get $\quad V^*: H^1_{dR}(E/Y_1(N)) \twoheadrightarrow \omega^{(p)}_{E/Y_1(N)} \cong \omega^{\otimes p}$

$\quad\quad \cup\!\!\!\mid$

$\quad\quad \omega$

Hasse invariant map

when pulling back a line bundle along Frobenius, the transition maps get p^{th}-powered, so $\omega^{\otimes p}$

$h := V^* \in \mathrm{Hom}_{\mathcal{O}_{Y_1(N)}}(\omega, \omega^{\otimes p}) = \Gamma(\mathcal{O}_{Y_1(N)}, \omega^\vee \otimes \omega^{\otimes p}) \cong \Gamma(\mathcal{O}_{Y_1(N)}, \omega^{p-1})$

This is called the Hasse invariant; it is a weight $p-1$ mod p modular form.

__Fact__: The q-expansion for h is just 1. __Fact__: h has no repeated zeros.

__Fact__: h is the reduction mod p of Eisenstein series E_{p-1}. (somewhat coincidental)

__Lemma__: The zero locus of h, $Z(h)$, is precisely the locus of $Y_1(N)$ where E is supersingular.

__Proof__: At a point $x \in Y_1(N)(\overline{\mathbb{F}}_p)$, the elliptic curve $E_{\bar{x}}$ is ordinary

$$\Rightarrow \quad E_{\bar{x}}[p] \cong \underset{\substack{\circlearrowleft \\ Fr_{E_{\bar{x}}/\bar{x}} = id}}{\mathbb{Z}/p\mathbb{Z}} \oplus \underset{\substack{\circlearrowleft \\ Fr_{E_{\bar{x}}/\bar{x}} = 0}}{\mu_p} \quad \text{as group scheme.}$$

$$\Rightarrow \quad V = 0 \qquad V \text{ is an isom.}$$

__Note__: $\omega_{E_{\bar{x}}/\bar{x}} \cong \omega_{E_{\bar{x}}[p]/\bar{x}} \circlearrowleft V$ is an isom

$\Rightarrow$ h doesn't vanish at this point.

Conversely, $V^* : \omega_{E_{\bar{x}}/\bar{x}} \longrightarrow \omega_{E_{\bar{x}}^{(p)}/\bar{x}}$ is an isom. $\Rightarrow$ $\operatorname{Ker} V$ is an étale group scheme

$\Rightarrow$ $E_{\bar{x}}$ must be ordinary. $\square$

__Remark__: On $X_1(1)$, with some care of the stackness, can count supersingular elliptic curves by understanding $\deg \omega^{p-1}$ (& Kodaira-Spencer $\omega^{\otimes 2} \simeq \Omega^1_X(\log C)$).

__Example__: (Fake Hilbert moduli variety)

F totally real field of discriminant D, $p \nmid ND$

$\Rightarrow$ p unram. in F.

Say $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_g$ & $[\mathbb{F}_{\mathfrak{p}_i} : \mathbb{F}_p] = f_i$

Take r to be divisible by all f_i

Fix an isom. $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$. then

$$\begin{array}{ccc} \mathcal{A} & \longleftarrow & A \\ \downarrow & & \downarrow \\ \mathcal{M} & \longleftarrow & M \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{Z}[\frac{1}{ND}] & \longleftarrow & \operatorname{Spec} \mathbb{F}_{p^r} \end{array} \quad \text{Special fiber.}$$

$$\operatorname{Hom}(F, \mathbb{C}) = \coprod_i \underbrace{\operatorname{Hom}(\mathcal{O}_{F_{\mathfrak{p}_i}}, \mathbb{Z}_{p^r})}$$

$\sigma = $ Frob on $\mathbb{Z}_{p^r}$

embeddings here are $\tau, \sigma \circ \tau, \sigma^2 \circ \tau, \ldots$

we denote them as $\tau_1^{(i)}, \tau_2^{(i)}, \ldots$ so that $\tau_{j+1}^{(i)} = \sigma \circ \tau_j^{(i)}$

& the subindices are modulo f_i.

Then $0 \longrightarrow \omega_{A/M} \longrightarrow H^1_{dR}(A/M) \longrightarrow \operatorname{Lie}_{A^\vee/M} \longrightarrow 0$

$$H^1_{dR}(A/M) \circlearrowleft \mathcal{O}_F \otimes \mathcal{O}_M \cong \bigoplus_{\mathfrak{p}_i} \bigoplus_{j=1}^{f_i} \mathcal{O}_M$$

Fix a $\mathfrak{p}_i$ for the rest of the discussion, get $0 \to \omega_{\tau_j} \to H^1_{dR,\tau_j} \to \mathrm{Lie}_{\tau_j} \to 0$

The Verschiebung map $V : H^1_{dR}(A/M) \to (\omega_{A/M})^{(p)}$

$$\Rightarrow \quad H^1_{dR,\tau_j} \twoheadrightarrow (\omega_{\tau_{j-1}})^{(p)} \simeq \omega_{\tau_{j-1}}^{\otimes p}$$

Note the twist from τ_j to τ_{j-1} here! $(\omega^{(p)})_{\tau_j} \cong (\omega_{\tau_{j-1}})^{(p)}$

$\rightsquigarrow$ partial Hasse invariant $h_j : \omega_{\tau_j} \longrightarrow \omega_{\tau_{j-1}}^{p}$

so $h_j \in H^1(M, \omega_{\tau_j}^{-1} \otimes \omega_{\tau_{j-1}}^{p})$ is a mod p Hilbert modular form of wt -1 at τ_j & p at τ_{j-1}

Note: h_j is not of paritious weight, so it can't be lifted to char 0. This is a purely char 0 phenomenon.

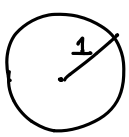
$h := \prod_{\mathfrak{p}_i} \prod_{j=1}^{f_i} h_j \in H^1(M, \otimes \omega_{\tau_j}^{p-1})$ is called the total Hasse invariant

Fact: The complement of the zero locus of h in M is the ordinary locus, i.e. where $A_{\bar{x}}[p] \simeq \mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_F \oplus \mu_p \otimes_{\mathbb{Z}} \mathcal{O}_F$ for $\bar{x} \in M^{ord}(\bar{\mathbb{F}}_p)$.

Remark: Our discussion crucially need p to be unramified in F.

§2 Overconvergent modular forms

Quick recall of some facts on rigid analytic space:

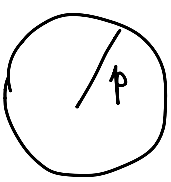
*  $D := D(0;1)$ = closed unit disc

$/\mathbb{Q}_p$ Ring of rigid analytic functions on D is

$$\mathbb{Q}_p\langle x\rangle := \{ f(x) = \sum_{n\geq 0} a_n x^n,\ a_n \in \mathbb{Q}_p,\ |a_n| \to 0 \text{ as } n \to +\infty \}$$

Gauss norm. $\|f(x)\| = \max_{n\geq 0} |a_n| = \sup_{z \in D(0;1)} |f(z)|$

Write $D = \mathrm{Spm}\, \mathbb{Q}_p\langle x\rangle$

 $D' = D(0;p)$ = closed disc of radius p

$$\leftrightarrow \quad \mathbb{Q}_p\langle px\rangle = \{ f(x) = \sum_{n\geq 0} a_n x^n,\ a_n \in \mathbb{Q}_p,\ p^n |a_n| \to 0 \text{ as } n \to +\infty \}$$

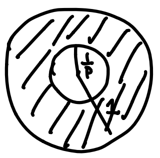
We can restrict functions on D' to that on D

i.e. $\mathbb{Q}_p\langle px\rangle \longrightarrow \mathbb{Q}_p\langle x\rangle$

$$\phi : \widehat{\bigoplus_{n\geq 0}} p^n \mathbb{Z}_p \cdot x^n \longrightarrow \widehat{\bigoplus_{n \geq 0}} \mathbb{Z}_p x^n$$

This is a p-adic limit of finite rank operators, i.e. ϕ_n = truncating at deg n.

So restricting strictly bigger space to strictly smaller space,
the map on rigid analytic fuctions is completely continuous/compact/limit of finite rank.



$$D^0 = D(0;1) - D(0;\tfrac{1}{p}^-) = \text{annulus of radius in } [\tfrac{1}{p}, 1]$$

$$\rightsquigarrow \quad \mathbb{Q}_p\langle x, \frac{p}{x}\rangle := \left\{ \sum_{n\in\mathbb{Z}} a_n x^n,\ a_n \in \mathbb{Q}_p \quad \begin{array}{l} |a_n| \to 0 \text{ as } n \to +\infty \\ |a_{-n}|\cdot p^n \to 0 \text{ as } n \to -\infty \end{array} \right\}$$

This is in fact the original definition

$$\overset{\text{}}{=} \left\{ \sum_{m\geq 0} f_m(x)\cdot\left(\frac{p}{x}\right)^m,\ f_m(x) \in \mathbb{Q}_p\langle x\rangle,\ \|f_m\| \to 0 \text{ as } m \to +\infty \right\}$$

rewrite this slightly

$$= \left(\bigcup_{m\geq 0} \left(\frac{p}{x}\right)^m \mathbb{Z}_p\langle x\rangle \right)^\wedge \left[\frac{1}{p}\right]$$

<u>Remark</u>: We can also write $D^0 = D(0;1) - (p^2; (\tfrac{1}{p})^-)$

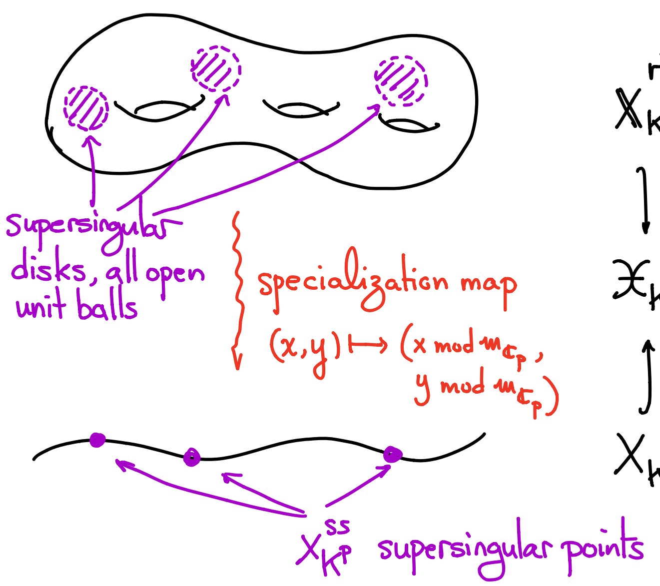
$$\rightsquigarrow \quad \mathbb{Q}_p\langle x, \frac{p}{x-p^2}\rangle = \left\{ \sum_{m\geq 0} f_m(x) \left(\frac{p}{x-p^2}\right)^m,\ f_m(x) \in \mathbb{Q}_p\langle x\rangle,\ \|f_m\| \to 0 \text{ as } m \to +\infty \right\}$$

but we can write $\frac{p}{x-p^2}$ as $\frac{p}{x}\cdot\frac{1}{1-\frac{p^2}{x}} = \frac{p}{x}\left(1 + \frac{p^2}{x} + \left(\frac{p^2}{x}\right)^2 + \cdots\right)$

to reduce to the previous expression.

Move to the case of modular curves $\mathfrak{X}_{K^p}$ (assuming the compactification is not a problem)

K^p open compact subgroup of $GL_2(\mathbb{A}_f^{(p)})$



$X_{K^p}^{rig} / \mathrm{Spm}\, \mathbb{Q}_p$ rigid analytic space

$\downarrow$

$\mathfrak{X}_{K^p} / \mathrm{Spf}\, \mathbb{Z}_p$

$\uparrow$

$X_{K^p} / \mathrm{Spec}\, \mathbb{F}_p$ special fiber

Why open disks: Since $\mathfrak{X}_{K^p}$ is a smooth curve over $\mathbb{Z}_p$, the completion of $\mathfrak{X}_{K^p}$ at a closed point $x \in X_{K^p}$

... over $\mathbb{Z}_p$, the completion of ... at a closed point ...

is $\mathrm{Spf}\left(W(k_x)[[\varpi_x]]\right)$, the one-variable formal power series ring.

$\updownarrow$

open unit disk, as ϖ_x must take values in $\mathfrak{m}_{\mathbb{C}_p}$.

Fix $0 < r < 1$ ($\& \ r \in p^{\mathbb{Q}}$)

Fix a lift $\tilde{h}$ of the Hasse invariant h. in $H^0(\mathcal{X}_{K^p}, \omega^{\otimes(p-1)})$

Then $\tilde{h}$ is a local parameter at each supersingular disk

Then $X_{K^p}^{rig} \setminus (\text{all supersingular open disks}) = \{ z \in X_{K^p}^{rig} \ ; \ |h(z)| = 1 \}$

Define $X_{K^p}^{rig}(r) := \{ z \in X_{K^p}^{rig} \ ; \ |h(z)| \geq r \}$

$= X_{K^p}^{rig} \setminus (\text{all supersingular disks of radius } r^-)$

Key: If $\mathcal{X}_{K^p}$ is defined over $\mathbb{Z}_p$ & thus h is, $X_{K^p}^{rig}(r)$ is well-defined for $\frac{1}{p} < r < 1$.

(See the explanation above with parameters x & $x - p^2$.)

<u>Definition</u> $S_k^{\dagger,r}(K^p) := H^0(X_{K^p}^{rig}(r), \omega^k)$ (better to assume $\frac{1}{p} < r < 1$)

space of <u>overconvergent modular forms</u> of wt k, level K^p & radius r.

Explicitly, $S_k^{\dagger,r}(K^p) = \left(\bigcup_{m \geq 0} \left(\frac{\lambda}{\tilde{h}}\right)^m H^0(\mathcal{X}_{K^p}, \omega^{k+(p-1)m}) \right)^{\wedge} [\frac{1}{p}]$

as $\tilde{h}$ is a section of ω^{p-1}, we need to twist ω^k to $\omega^{k+(p-1)m}$.

Compare with our earlier example

$S_k^{\dagger}(K^p) := \bigcup_{r \to 1^-} S_k^{\dagger,r} \ K^p$

§3 Canonical subgroups and U_p-operator

Let S be a scheme of char $p > 0$, e.g. $S = \mathcal{O}_{\mathbb{C}_p}/(p)$.

Let $A \xrightarrow{\pi} S$ be an abelian variety, $\omega_{A/S} := \Lambda^{\dim_S A}(\pi_* \Omega^1_{A/S})$

Verschiebung $V : A^{(p)} \longrightarrow A$

$\rightsquigarrow V^* : \omega_{A/S} \longrightarrow \omega_{A^{(p)}/S} \cong \omega_{A/S}^p$

Hasse invariant $Ha(A/S) \in H^0(S, \omega_{A/S}^{p-1})$

Black Box Theorem Let R be a p-adically complete flat $\mathbb{Z}_p^{cycl}$-alg (e.g. $\mathcal{O}_{\mathbb{C}_p}$) & A/R an abelian var.

(1) $\mathrm{Ha}(A\otimes R/p)$ is a unit in $R/p \Rightarrow A\otimes R/p$ is ordinary

(2) Assume $\mathrm{Ha}(A\otimes R/p)^{\frac{p^m-1}{p-1}} \in R/p$ divides p^{ε} for $\varepsilon < \frac{1}{2}$.

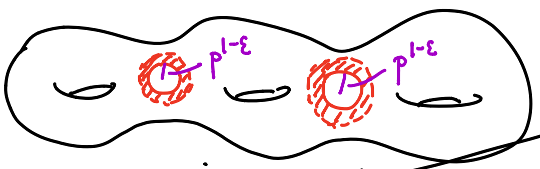
Then there exists a unique closed subgroup $C \subset A[p^m]$ such that $C = \ker \mathrm{Fr}^m \bmod p^{1-\varepsilon}$

i.e.
$$\begin{array}{ccc} A[p^m] \supseteq \operatorname{Ker} \mathrm{Fr}^m & & C \\ \Big| & \xrightarrow{\text{lifts uniquely}} & \Big| \\ R/p^{1-\varepsilon} & & R \end{array}$$

* In the ordinary case, $A/\mathcal{O}_{\mathbb{C}_p}$, s.t. $A\otimes\overline{\mathbb{F}}_p$ ordinary

$A[p^\infty] \simeq \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p$ & the canonical subgp is $\mu_{p^m} = A[p^m]^{\mathrm{conn.}}$

Back to our situation at hand:



i.e. where $\mathrm{Ha}(E) \mid p^{\varepsilon}$

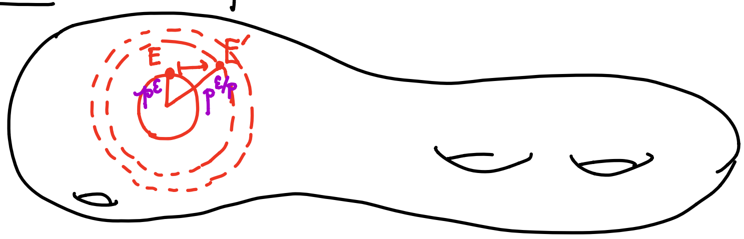
So over $X^{\mathrm{rig}}_{K^p}(p^{\varepsilon})$, there exists a canonical subgroup $C \subseteq E[p]$ of order p.

Define a U_p-operator on $S_k^{\dagger, p^{\varepsilon}}(K^p)$ (in terms of Katz modular form)

$$U_p(f)(E, i, \omega) = p^{k-1} \sum_{\substack{C' \subset E[p] \\ C' \neq C}} f(E/C', i_{C'}, \check{\pi}^* \omega)$$

Here, we only take objects (E, i) over $X^{\mathrm{rig}}_{K^p}(p^{\varepsilon})$

Important claim: The "map" $E \longmapsto E' = E/C'$ looks like when $\varepsilon < \frac{1}{2}$



So the U_p-operator factors as $S_k^{\dagger, p^{\varepsilon}}(K^p) \xrightarrow{\text{res}} S^{\dagger, p^{\varepsilon/p}}(K^p) \xrightarrow{U_p} S^{\dagger, p^{\varepsilon}}(K^p)$

Compact operator! b/c $X^{\mathrm{rig}}_{K^p}(p^{\varepsilon/p})$ is strictly contained in $X^{\mathrm{rig}}_{K^p}(p^{\varepsilon})$

Continuous

$\Rightarrow$ U_p-operator is compact

$\Rightarrow$ Given any $\lambda > 0$, only finitely many U_p-eigenvalues are $\geq \lambda$.

Proof of the claim:

Note $E'/\mathrm{Im}(C \to E') = E/C+C' = E/E[p] \simeq E$

& $C \bmod p^{1-\varepsilon} = \mathrm{Ker}\, Fr_{E \bmod p^{1-\varepsilon}} \Rightarrow \mathrm{Im}(C \to E') \bmod p^{1-\varepsilon} = \mathrm{Ker}\, Fr_{E' \bmod p^{1-\varepsilon}}$

$\Rightarrow E \bmod p^{1-\varepsilon} \cong (E' \bmod p^{1-\varepsilon})^{(p)}$

$\Rightarrow Ha(E \bmod p^{1-\varepsilon}) = Ha(E' \bmod p^{1-\varepsilon})^{p}$

So if $Ha(E \bmod p^{1-\varepsilon}) = p^{\varepsilon} \cdot \text{unit}$, then $Ha(E' \bmod p^{1-\varepsilon}) = p^{\varepsilon/p} \cdot \text{unit}$