Arithmetic compactification of Hilbert modular varieties

§1. Degeneration of abelian varieties

Ref: Mumford, An analytic construction of degenerating abelian varieties over complete rings.

Writing style: Example / Prototype, usually / $\mathbb{C}$. Main logic flow Comments

Recall: (Over $\mathbb{C}$). To construct an abelian variety, we need

$$\Lambda \simeq \mathbb{Z}^{\oplus 2n}, \quad E: \Lambda \times \Lambda \longrightarrow \mathbb{Z} \text{ alternating form}$$

& a complex structure on $\Lambda_{\mathbb{R}}$ s.t. $E(iu, iv) = E(u,v)$ & $E(u, iu) > 0 \ \forall u \neq 0$.

Take $\underbrace{X^\vee}$ $\subseteq \mathbb{Z}^{\oplus 2n}$ a Lagrangian lattice, i.e. $X^\vee$ saturated, rank n, $E|_{X^\vee \times X^\vee} = 0$

To be consistent w/ later notation, $X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$

Choose $\mathbb{Z}^{\oplus 2n} = X^\vee \oplus Y$ so that Y is also Lagrangian

Identify $\Lambda_{\mathbb{R}} \simeq X^\vee_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} X^\vee$, then

$$\Lambda_{\mathbb{R}} / \Lambda \cong \mathbb{C} \otimes_{\mathbb{Z}} X^\vee / X^\vee \oplus Y \xrightarrow[\exp(2\pi i \cdot)]{\simeq} X^\vee \otimes_{\mathbb{Z}} \mathbb{C}^\times / \tilde{q}^Y \qquad \text{where } \tilde{q}^Y = \{ e^{2\pi i y} \mid y \in Y \}$$

Goal: Construct $X^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m / \tilde{q}^Y$ in "family"

Let R be an excellent integrally closed noetherian ring, $\leftarrow$ Typical example in $R = \mathbb{Z}_p[[q]]$, $I = (q)$.

$I = \sqrt{I} \subseteq R$ ideal. $K = \mathrm{Frac}(R)$. $\qquad S = \mathrm{Spec}\, R$, $S_0 = \mathrm{Spec}\, R/I$

Assume that R is complete for I-adic topology.

Consider an algebraic torus $\tilde{\mathcal{G}}$ over R. i.e. $\tilde{\mathcal{G}} \simeq \mathbb{G}_{m,R}^n$

More canonically, set $X := X^\bullet(\tilde{\mathcal{G}}) := \mathrm{Hom}_R(\tilde{\mathcal{G}}, \mathbb{G}_{m,R}^n) \simeq \mathbb{Z}^{\oplus n}$

thus $\tilde{\mathcal{G}} = \mathbb{G}_{m,R} \otimes_{\mathbb{Z}} X^\vee$ $\leftarrow$ Serre Tensor, where $X^\vee := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$

$$= \underbrace{R[X]}_{\text{group algebra}} = R[\chi^\alpha; \alpha \in X] / \chi^{\alpha+\beta} = \chi^\alpha \cdot \chi^\beta$$

In particular, χ^α is a function on $\tilde{\mathcal{G}}$

Consider a subgroup $Y \subseteq \tilde{\mathcal{G}}(K)$ isomorphic to $\mathbb{Z}^{\oplus n}$

$\exists$ a natural pairing $q: X \times Y \xrightarrow{*} K^\times$

$\alpha, y \mapsto \mathcal{X}^\alpha(y) \in K^\times$

or equivalently
$\alpha: \tilde{\mathcal{G}} \longrightarrow \mathbb{G}_m$
$y \longmapsto \mathcal{X}^\alpha(y)$

or equivalently $\tilde{q}: Y \longrightarrow X^\vee \otimes \mathbb{G}_{m,K}$

"Goal" Construct $\tilde{\mathcal{G}}/Y \approx \mathbb{G}_{m,R} \otimes_{\mathbb{Z}} X^\vee / Y$.

<u>Def'n</u>: A <u>polarization</u> for the period Y is a homomorphism $\phi: Y \to X$

s.t. (1) $\mathcal{X}^{\phi(y)}(z) = \mathcal{X}^{\phi(z)}(y)$ for all $y, z \in Y$

(2) $\mathcal{X}^{\phi(y)}(y) \in I \leftarrow$ the ideal for all $y \in Y$, $y \neq 0$

($\Rightarrow \phi$ is injective & $[X : \phi Y] < \infty$, & some usual positivity.)

Let's explain where (1) & (2) comes from.

$E: \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ induces $\tilde{\phi}: X^\vee \times \overset{Y}{\overset{\|}{\Lambda / X^\vee}} \longrightarrow \mathbb{Z}$

<u>Note</u>: $\mathcal{X}^{\phi(y)}(z) = q(\phi(y), z) = \tilde{\phi}_{\mathbb{C}^\times}(\tilde{q}(y), z)$

<u>Exercise</u>: Condition $E(iu, iv) = E(u, v)$ implies (1)

The other condition $\longleftrightarrow$ (2)

<u>Goal</u>: Construct a semiabelian variety $\mathcal{G}$ on S s.t.

(1) $\mathcal{G}_\eta$ is an abelian variety of dim n & "$\mathcal{G}_\eta = \tilde{\mathcal{G}}/Y$"

(2) $\mathcal{G} \times_S S_0$ is a split torus $\mathbb{G}^n_{m, S_0}$.

E.g. "$\mathbb{G}_m / q^{\mathbb{Z}}$" over $\mathbb{C}[[q]]$. at $q = 0$, we see "$\mathbb{G}_m$".

<u>Step 1</u>: Construct a "line bundle" that is supposedly the pull back of ample line bundle on "$\tilde{\mathcal{G}}/Y$" defined by ϕ

<u>Back to $\mathbb{C}$-story</u>:

$$\begin{tikzcd} \mathbb{C}^n \times \mathbb{C} \arrow[d] & \mathcal{L}_E \arrow[d, dash] \\ \mathbb{C}^n \arrow[r] & \mathbb{C}^n/\Lambda \end{tikzcd}$$

described by Appel-Humbert theorem.

For us, we'll pull back to $(\mathbb{C}^\times)^n \cong X^\vee \otimes \mathbb{C}^\times$ & study transition function there.

Recall: $X^\vee \otimes \mathbb{G}_{m,R} = \operatorname{Spec} R[\mathcal{X}^\alpha ; \alpha \in X^\vee] / (\mathcal{X}^\alpha \cdot \mathcal{X}^\beta = \mathcal{X}^{\alpha+\beta}, \mathcal{X}^0 = 1)$

Consider $\mathcal{R} := R[\mathcal{X}^\alpha ; \alpha \in X^\vee] / (\mathcal{X} \cdots\cdots)[\theta]$

This carries an action of Y by: for $y \in Y$, $S_y^* : \mathcal{R} \to \mathcal{R}$

$S_y^*(a) = a$ for $a \in R$

$S_y^*(\mathcal{X}^\alpha) = \mathcal{X}^\alpha(y) \cdot \mathcal{X}^\alpha$ (because y acts by translation by $\tilde{q}(y)$)

$S_y^*(\theta) = \mathcal{X}^{\phi(y)}(y) \cdot \mathcal{X}^{2\phi(y)} \cdot \theta.$

(We explain that this corresponds to the line bundle with $4E$.)

Recall from Mumford's AV book: pulling back the line bundle $[4E]$ to $X^\vee \otimes \mathbb{C} \simeq \mathbb{C}^n$

the action of Λ on $(X^\vee \otimes \mathbb{C}) \times \mathbb{C}$ is given by, for $u \in \Lambda$,

$$u \cdot (z, \lambda) = \left(z+u, \alpha(u) \cdot e^{4\pi H(z,u) + 2\pi H(u,u)}\right)$$

where $\alpha : \Lambda \to S^1$ satisfies $\alpha(u_1+u_2) = \alpha(u_1) \cdot \alpha(u_2) \cdot e^{4\pi i E(u_1,u_2)}$

We need to modify this action so that $X^\vee \subseteq \Lambda$ acts trivially.

Following §3 therein, as $E|_{X^\vee \times X^\vee} = 0 = E|_{Y \times Y} \Rightarrow \exists$ char $\alpha' : \Lambda \to S^1$ s.t. $\alpha'(x) = \alpha(x)$ $\forall x \in X^\vee$ or $x \in Y$.

we change the section by a scalar $z \mapsto e^{2\pi B(z,z)} \cdot \alpha'(z)^{-1}$

where $B : X^\vee_{\mathbb{C}} \times X^\vee_{\mathbb{C}} \to \mathbb{C}$ is a <u>bilinear</u> symmetric form s.t. $B|_{X^\vee \times X^\vee} = H|_{X^\vee \times X^\vee}$

$\rightsquigarrow$ get a new action $u \cdot (z, \lambda) = \left(z+u, e^{4\pi (H-B)(z,u) + 2\pi (H-B)(u,u)}\right)$ for $u \in X^\vee$ or Y.

But then the condition $\Rightarrow X^\vee$ acts trivially.

<u>Exercise</u>: This agrees w/ the action above.

<u>Idea</u>: Take the "minimal" ring R_ϕ on which this action can be defined.

<u>Key Example</u>: $R = \mathbb{C}[\![q]\!]$, $I = (q)$, $X^\vee = \mathbb{Z}$, $Y \hookrightarrow (X^\vee \otimes \mathbb{G}_m)(K)$

$\phi : Y \to X$ identity $\qquad \mathbb{Z} \ni 1 \longmapsto q$

Then $S = S_1 \curvearrowright \mathcal{R} = \underbrace{\mathbb{C}((q))[\mathcal{X}^{\pm 1}]}_{\mathbb{G}_{m,R}}[\theta]$

$S(a)=a$ for $a\in\mathbb{C}[[q]]$, $S(X)=qX$, $S(\theta)=q\cdot X^2\theta$

Consider certain partial compactification of bundles on $\mathbb{G}_m$.

Take $R_\phi=\mathbb{C}[[q]]\left[\begin{matrix}\theta,\ S(\theta)=q\cdot X^2\theta,\ \dots,\ S_k(\theta)=q^{k^2}X^{2k}\theta, \\ X\theta,\ S(X\theta)=q^2X^3\theta,\ \dots,\ S_k(X\theta)=q^{k^2+k}X^{2k+1}\theta\end{matrix}\ ;\ k\in\mathbb{Z}\right]$

$\uparrow\ S^{-1}(X\theta)=X^{-1}\theta$ b/c $S(X^{-1}\theta)=q^{-1}X^{-1}\cdot qX^2\theta$

Viewing R_ϕ as a graded algebra $\deg\theta=1$. $\leftarrow$ NOT finitely generated!

$\rightsquigarrow \mathrm{Proj}(R_\phi) =: \tilde{P} \leftarrow\cdots \tilde{\mathcal{G}}$ in fact open immersion

natural map b/c $R_\phi\subseteq\mathbb{C}[[q]][[X^{\pm1}]][\theta]$

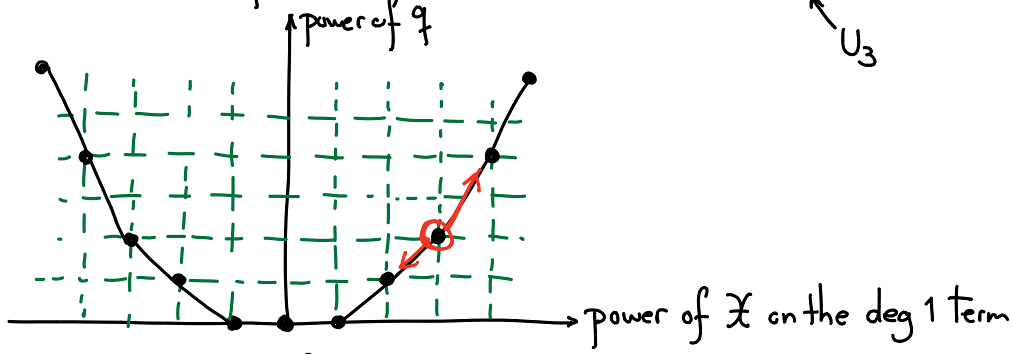
<u>Note</u>: $\mathrm{Proj}(R_\phi)=\bigcup_{f\in\deg 1\text{ generators}}\mathrm{Spec}\left(R_\phi[\tfrac{1}{f}]^{\deg=0}\right)$

U_0

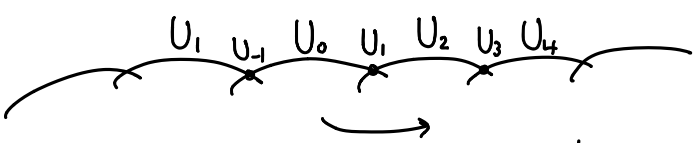
$R_\phi[\frac{1}{\theta}]^{\deg=0}=\mathbb{C}[[q]]\left[\begin{matrix}1,\ qX^2,\ \dots,\ q^{k^2}X^{2k} \\ X,\ X^{-1},\ q^2X^3,\ \dots,\ q^{k^2+k}X^{2k+1}\end{matrix}\right]=\mathbb{C}[[q]][X,X^{-1}] \longleftrightarrow |X|=1$ annulus

$R_\phi[\frac{1}{q^2X^3\theta}]^{\deg=0}=\mathbb{C}[[q]]\left[\begin{matrix}q^{-2}X^{-3},\ q^{-1}X^{-1},\ q^2X \\ q^{-2}X^{-2},\ \dots\end{matrix}\right]=\mathbb{C}[[q]][q^{-1}X^{-1},q^2X] \longleftrightarrow |X|\in[q,q^2]$

U_3

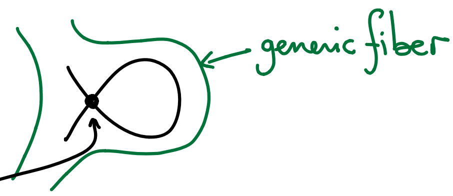


In other words, the special fiber of Proj R_ϕ looks like



S-action, translation by 2.

Taking "quotient" by S-action $\Rightarrow$ $P=$



$G:=P$ removing $\bullet$

more calculation of this in the exercise.

Now, more rigorously and more generally

we choose a subset $\Sigma\subseteq X$ s.t. $0\in\Sigma$ and $\Sigma=-\Sigma$ and Σ contains a basis of X.

Define $R_{\phi,\Sigma} := R\left[S_y(\mathcal{X}^\alpha \theta) \;;\; \begin{array}{l}\text{for all } y\in Y \\ \text{for all } \alpha\in\Sigma\end{array}\right]$

$$= R\left[\cdots, \mathcal{X}^\alpha\theta,\ \mathcal{X}^{\phi(y)+\alpha}(y)\cdot \mathcal{X}^{2\phi(y)+\alpha}\cdot\theta, \cdots \;;\; \begin{array}{l}\text{for all } y\in Y \\ \text{for all } \alpha\in\Sigma\end{array}\right]$$

(Take ϕ "positive enough" so that $\mathcal{X}^{\phi(y)+\alpha}(y)\in A \;\; \forall y\in Y, \alpha\in\Sigma$.)

$$\big\downarrow$$

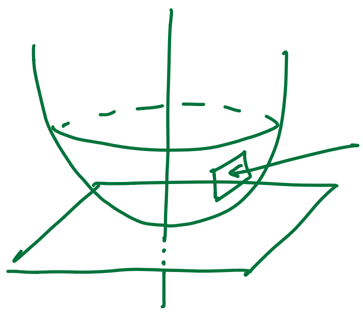
$$R\left[\mathcal{X}^\alpha\theta, \cdots ; \text{for all } \alpha\in X\right]$$

Take $\tilde{P} := \operatorname{Proj} R_{\phi,\Sigma} \xhookleftarrow[\text{open.}]{} \operatorname{Proj}\left(R[\mathcal{X}^\alpha\theta, \cdots, \text{for all } \alpha\in X\right) = \tilde{\mathcal{Y}}$

Localizing at all degree 1 generators of $R_{\phi,\Sigma}$, we get

$$\tilde{P} = \bigcup_{\substack{\alpha\in\Sigma \\ y\in Y}} U_{\alpha,y} \quad \text{with } U_{\alpha,y} = \operatorname{Spec} R\left[\frac{\mathcal{X}^{\phi(z)+\beta}(z)\cdot\mathcal{X}^{2\phi(z)+\beta}}{\mathcal{X}^{\phi(y)+\alpha}(y)\cdot\mathcal{X}^{2\phi(y)+\alpha}},\ \begin{array}{l}\text{for all } z\in Y \\ \text{all } \beta\in\Sigma\end{array}\right]$$

If we visualize this as above, we get



- Each $U_{\alpha,y}$ is of finite type
- $\forall y'\in Y$, $S_{y'}(U_{\alpha,y}) = U_{\alpha,y+y'}$

<u>Step 2</u>. Perform the quotient by Y.

* <u>Need to be careful</u>: $\tilde{G}_\eta \simeq \tilde{P}_\eta$

b/c $$K\left[\frac{\mathcal{X}^{\phi(z)+\beta}(z)}{\mathcal{X}^{\phi(y)+\alpha}(y)}\cdot\frac{\mathcal{X}^{2\phi(z)+\beta}}{\mathcal{X}^{2\phi(y)+\alpha}},\ \begin{array}{l}\text{for all } z\in Y \\ \text{all } \beta\in\Sigma\end{array}\right] = K\left[\mathcal{X}^\alpha ; \alpha\in X\right] = \mathcal{O}_{\tilde{G}_\eta}$$

in K (but not in R)

So these $U_{\alpha,y}$'s have different special fibers but <u>same</u> generic fiber!

- Let $\hat{U_{\alpha,y}}$ denote the completion of $U_{\alpha,y}$ along its special fiber $U_{\alpha,y}\times_R R/I$.

<u>Fact</u>: $\forall \alpha, y$, there's only finitely many other β, z s.t. $\hat{U_{\alpha,y}} \cap \hat{U_{\beta,z}} \neq \phi$.

$\rightsquigarrow P :=$ quot of completion of $\tilde{P}$ along its special fiber

Further arguments : $\tilde{P}$ carries a natural ample line bundle $\tilde{\mathcal{L}} = \tilde{\mathcal{O}}(1)$

* Show that $\tilde{\mathcal{O}}(1)$ can be quotiented as well to get $\mathcal{L}$ ample on P
* Use Grothendieck's formal algebraization to make P algebraic.
* Define $G := P - \text{“}(\tilde{P} - \tilde{G})/Y\text{”}$ & show that G is a semi-abelian scheme.

Details are in Mumford's paper.

§2 Compactification of Hilbert modular variety

Reference: C.L.-Chai, Arithmetic minimal compactification of the Hilbert-Blumenthal moduli spaces

- F a totally real field. δ = different ideal.
- Will work with $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$. $\Gamma(N)$-level structure

① Moduli problem

Fix a set $\{\mathfrak{c}_1, \cdots, \mathfrak{c}_{h^+}\}$ a set of representatives of the strict ideal class group of F.

For $\mathfrak{c} = \mathfrak{c}_i$, set $M_N(\mathfrak{c})$ = moduli space over $\mathbb{Z}[\zeta_N, \frac{1}{N}]$,

sending a $\mathbb{Z}[\zeta_N, \frac{1}{N}]$-scheme S to isomorphism classes of (A, λ, η)

- A is an abelian scheme over S of dim $[F:\mathbb{Q}]$, equipped with faithful $\mathcal{O}_F$-action
- $\mathcal{O}_F$-linear $\lambda : A \otimes \mathfrak{c} \xrightarrow{\sim} A^\vee$ s.t. the

$$\begin{tikzcd}
\mathfrak{c} \arrow[r] \arrow[d, phantom, "\cup"] \arrow[drr, "\simeq"] & \mathrm{Hom}_{\mathcal{O}_F}(A, A\otimes\mathfrak{c}) \arrow[r, "\lambda", "\sim"'] & \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee) \arrow[d, phantom, "\cup"] \\
\mathfrak{c}^+ \arrow[drr, "\simeq"] & & \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)^{\mathrm{Sym}} \arrow[d, phantom, "\cup"] \\
 & & \mathrm{Pol}_{\mathcal{O}_F}(A, A^\vee)
\end{tikzcd}$$

- $\eta : (\mathcal{O}_F / N\mathcal{O}_F)^{\oplus 2} \xrightarrow{\sim} A[N]$ $\mathcal{O}_F$-linear isom.

(This induces an isom. $\mathcal{O}_F/N\mathcal{O}_F \xrightarrow{\sim} \wedge^2_{\mathcal{O}_F} A[N] \xrightarrow[\cong]{\lambda} \mathfrak{c}^{-1}\delta^{-1} \otimes_{\mathbb{Z}} \mu_N$.)

$M_N := \coprod_i M_N(\mathfrak{c}_i)$

② Cusp labels

comes with a natural positivity

Definition. A cusp label of $M(\mathfrak{c})$ consists of the following data

(i) projective rank-one modules $\mathfrak{a}, \mathfrak{b}$

(ii) an exact sequence of projective $\mathcal{O}_F$-modules $0 \to \delta^{-1}\mathfrak{a}^{-1} \to H \to \mathfrak{b} \to 0$

(iii) an isomorphism $\beta : \mathfrak{b}^{-1}\mathfrak{a} \xrightarrow{\sim} \mathfrak{c}$.

(Number of such isomorphism classes is $\# Cl(\mathcal{O}_F)$.)

Where does this come from? $GL_2(F) \backslash (\mathfrak{h}^{\pm})^{[F:\mathbb{Q}]} \times GL_2(\mathbb{A}_f) / GL_2(\hat{\mathcal{O}}_F) = GL_2(F)^{>0} \backslash \mathfrak{h}^{[F:\mathbb{Q}]} \times GL_2(\mathbb{A}_f) / GL_2(\hat{\mathcal{O}}_F)$

cusps are labeled by $GL_2(F) \backslash \mathbb{P}^1(F) \times GL_2(\mathbb{A}_{F,f}) / GL_2(\mathcal{O}_F)$

By Iwasawa decomposition, $GL_2(\mathbb{A}_{F,f}) = B(\mathbb{A}_{F,f}) \cdot GL_2(\hat{\mathcal{O}}_F)$

$\Rightarrow GL_2(F)_{>0} \backslash \mathbb{P}^1(F) \times GL_2(\mathbb{A}_{F,f}) / GL_2(\hat{\mathcal{O}}_F) = B(F)_{>0} \backslash B(\mathbb{A}_{F,f}) / B(\hat{\mathcal{O}}_F)$

As F is dense in $\mathbb{A}_{F,f}$ & $\mathbb{A}_{F,f}^{\times} / F^{\times}\hat{\mathcal{O}}_F^{\times} \simeq cl(F)$.

<u>Definition</u> A <u>cusp label for $M_N(\mathfrak{c})$</u> is a cusp label for $M_1(\mathfrak{c})$ as above together with an $\mathcal{O}$-linear isom. $\gamma : (\mathcal{O}_F / N\mathcal{O}_F)^2 \xrightarrow{\sim} H / N \cdot H$.

③ Periods for Hilbert-Tate abelian varieties.

Write $M := \mathfrak{a}\mathfrak{b} = \mathfrak{c}\mathfrak{b}^2$ & let M^+ denote the subset of totally non-negative elements

Let $R := \mathbb{Z}[\frac{1}{N}, \zeta_N][[q^{\alpha} ; \alpha \in M^+]] / (q^{\alpha} \cdot q^{\beta} = q^{\alpha+\beta}, q^0 = 1)$

In the way we set things up, we want to make the quotient

$$q : \mathfrak{b} \longrightarrow \left(\mathbb{G}_m \otimes_{\mathbb{Z}} (\mathfrak{a}^{-1}\delta^{-1}) \right)_R \quad \text{over some period ring } R.$$

$$\Updownarrow$$

$$\tilde{q} : M = \mathfrak{a}\mathfrak{b} \longrightarrow \mathbb{G}_m(R).$$

$$\alpha \longmapsto q^{\alpha}$$

$Y = \mathfrak{b}$

$X^{\vee} = \mathfrak{a}^{-1}\delta^{-1}$

$\Downarrow$

$X = \mathfrak{a}$.

Why do we want $R = \mathbb{Z}[\frac{1}{N}, \zeta_N][[q^{M^+}]]$ with the positivity?

The polarization is a map $\phi : Y \to X \rightsquigarrow \phi = \beta : \mathfrak{b} \otimes \mathfrak{c} \xrightarrow{\sim} \mathfrak{a}$.

We need some "positivity" on R, namely, $\tilde{q}(y, \phi(y)) \in I \subseteq R$ for $y \overset{\#0}{\in} Y = \mathfrak{b}$

* Quick digression on toric variety:

A <u>toric variety</u> over a field k is a variety X containing a torus $\mathbb{G}_m^n/k$ as an open subscheme and the $\mathbb{G}_m^n$-action on this torus extends to X.

E.g. $\mathbb{G}_m^n = \operatorname{Spec} k[x_1^{\pm1}, \dots, x_n^{\pm1}] = \operatorname{Spec} k[\mathbb{Z}^n]$

$$\downarrow$$

$\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n] = \operatorname{Spec} k[\mathbb{Z}^n_{\geq 0}]$

So basically, there's a <u>contravariant</u> functor

$$\{\text{submonoids of } \mathbb{Z}^n\} \longrightarrow \text{Schemes}/k$$
$$\sigma \longmapsto \operatorname{Spec} k[\sigma]$$

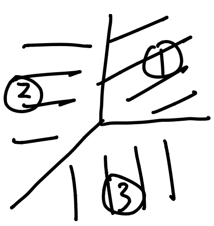
Better to make it covariant theory:

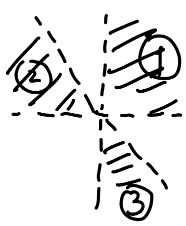
$$\{\text{Cones } \check{\sigma} \text{ in } (\mathbb{Z}^{n\vee})_{\mathbb{R}}\} \longrightarrow \{\text{submonoids of } \mathbb{Z}^n\}^{op} \longrightarrow \text{Sch}/k$$
$$\check{\sigma} \longmapsto \sigma := \{x \in \mathbb{Z}^n, \text{ s.t. } \langle x, y\rangle \geq 0 \ \forall y \in \check{\sigma}\} \longmapsto \operatorname{Spec} k[\sigma]$$

<u>E.g.</u>

$$\begin{array}{ccccc}
\check{\sigma} = (\mathbb{Z}^n_{\geq 0})^\vee \subseteq (\mathbb{Z}^n)^\vee & \longmapsto & \mathbb{Z}^n_{\geq 0} & \longmapsto & \operatorname{Spec} k[x_1, \dots, x_n] \\
\cup\!\text{I facet} & & & & \cup\!\text{I open subscheme} \\
(\mathbb{Z}^{n-1}_{\geq 0})^\vee \times \{0\} & \longmapsto & \mathbb{Z}^{n-1}_{\geq 0} \times \mathbb{Z} & \longmapsto & \operatorname{Spec} k[x_1, \dots, x_{n-1}, x_n^{\pm1}] \\
\cup & & & & \cup\!\text{I open torus} \\
\{0\} & \longmapsto & \mathbb{Z}^n & \longmapsto & \operatorname{Spec} k[x_1^{\pm1}, \dots, x_n^{\pm1}]
\end{array}$$

E.g.





$$\operatorname{Spec} k[x^{-1}, \tfrac{y}{x}] \underset{\operatorname{Spec} k[y, x^{\pm1}]}{\cup} \operatorname{Spec} k[x, y]$$

$$\underset{\operatorname{Spec} k[\frac{x}{y}, \frac{y}{x}, y^{-1}]}{\cup} \qquad \underset{\operatorname{Spec} k[x, y^{\pm1}]}{\cup}$$

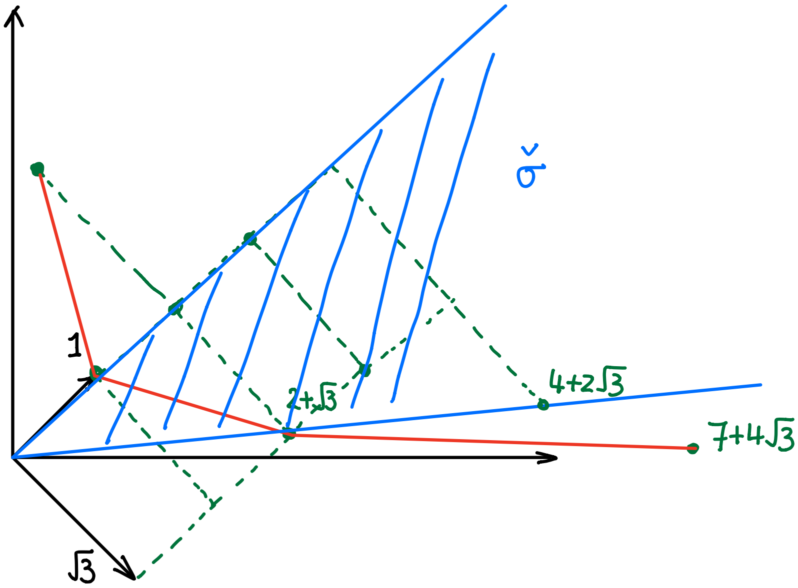
$$\operatorname{Spec} k[y^{-1}, xy^{-1}]$$

$$\| \\ \mathbb{P}^2$$

④ <u>Cone decomposition</u>

fundamental unit

E.g. $F=\mathbb{Q}(\sqrt{3})$, $\mathfrak{a}=\mathfrak{b}=(1)$ $\quad M^* = (\mathfrak{a}\mathfrak{b})^* \cong \mathbb{Z}[\sqrt{3}] \supseteq M^+ \ni u = 2+\sqrt{3}$



(a) $\mathbb{Z}[M^+]$ is <u>not</u> finitely generated.

(b) $\mathcal{O}_{F,N}^\times$ acts on $(\mathbb{G}_m \otimes \mathfrak{a}^{-1}\delta^{-1})/q^{\mathfrak{b}}$ by isom. preserving N-torsion points

$\rightsquigarrow (\mathcal{O}_{F,N}^\times)^2$ acts on $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![M^+]\!]$

& we are supposed to take $\operatorname{Spec}\left(\mathbb{Z}[\frac{1}{N}, \zeta_N][\![M^+]\!]\right) / (\mathcal{O}_{F,N}^\times)^2$

In practice, $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}, \zeta_N][\![\check{\sigma}]\!]$ as shown can be used as a proxy of

(In general, we need to discuss toric varieties here,

taking a cone decomposition of $M^{*,+}$ that is $\mathcal{O}_{F,N}^\times$-stable & then glue.)

Then we apply Mumford's construction to $(\mathbb{G}_m \otimes \mathfrak{a}^{-1}\delta^{-1})/q^{\mathfrak{b}}$ over $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![\check{\sigma}]\!]$

$\rightsquigarrow \mathcal{G}$ a degeneration of HBAV into $\mathbb{G}_m \otimes \mathfrak{a}^{-1}\delta^{-1}$ over $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![\check{\sigma}]\!]$.

<u>Theorem</u>: Upon fixing an $\mathcal{O}_{F,N}^\times$-stable rational polyhedral cone decomposition $\{\sigma\}$ of $M^{*,+}$,

there's a unique compactification $\mathcal{M}_N(\mathfrak{c}) \subseteq \mathcal{M}_N(\mathfrak{c})\{\sigma\}$

s.t. (1) the universal abelian variety A extends to a semiabelian variety G on $\mathcal{M}_N(\mathfrak{c})\{\sigma\}$

(2) there's an isom. $\displaystyle\coprod_{\substack{\text{cusp}\\\text{labels}}} \operatorname{Spec} \mathbb{Z}[\tfrac{1}{N}, \zeta_N][\![\check{\sigma}]\!] \xrightarrow{\sim} \mathcal{M}_N(\mathfrak{c})\{\sigma\}^{\wedge}$ ← completion along cusps

matching the boundary & natural universal family of abelian varieties.