GENERAL THEORY OF SHIMURA VARIETIES

Contents

1. Shimura Data

1.1. Definition and Examples.

Definition 1.1. A *Cartan involution* on a linear algebraic group G over $\mathbb{R}$ is an automorphism θ of G over $\mathbb{R}$ such that

(1) $\theta^2 = 1$;

(2) $\mathrm{G}^{(\theta)}(\mathbb{R}) := \{g \in \mathrm{G}(\mathbb{C}) \mid \theta(g) = \overline{g}\}$ is compact, where $\overline{g}$ is the complex conjugation of g.

Here is an example.

Example 1.2. Let $\mathrm{G} = \mathrm{SL}_2$ and $\theta = \mathrm{ad}(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$.

For every $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, we have

$$\theta(g) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right).$$

Thus $\mathrm{G}^{(\theta)}(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}) \mid d = \overline{a}, c = -\overline{b}\} = \{\left(\begin{smallmatrix} a & b \\ -\overline{b} & \overline{a} \end{smallmatrix}\right) \mid |a|^2 + |b|^2 = 1\}$.

In fact, we have the following theorem.

Theorem 1.3. *A linear algebraic group* G *over* $\mathbb{R}$ *has a Cartan involution if and only if* G *is reductive. In this case, two Cartan involutions differ from a conjugation by an element in* $\mathrm{G}(\mathbb{R})$.

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Fact 1.4. *Identity is a Cartan involution on* G *if and only if* G($\mathbb{R}$) *is compact.*

Definition 1.5. A *shimura datum* consists of a pair (G, X) where

($*$) G is a reductive algebraic group G over $\mathbb{Q}$ and

($**$) X is a G($\mathbb{R}$)-conjugacy class of homomorphisms $h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to \mathrm{G}_{\mathbb{R}}$ such that for one (hence for every) h, following conditions hold.

($SV1$) The composition $\mathbb{S} \xrightarrow{h} \mathrm{G}_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(\mathrm{G})_{\mathbb{R}})$ defines a Hodge structure on $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$ of types $(1,-1), (0,0), (-1,1)$. In other words, for every $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$, it acts on $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$ with eigenvalues $\frac{\overline{z}}{z}, 1, \frac{z}{\overline{z}}$.

($SV2$) Conjugation by the image of $h(i)$ in $\mathrm{G}^{\mathrm{ad}} = \mathrm{G}/\mathrm{Z}(\mathrm{G})$ is a Cartan involution.

($SV3$) For every $\mathbb{Q}$-simple factor H of G^{ad}, the group H($\mathbb{R}$) is not compact.

Remark 1.1. (1) By definition, we have $X \simeq \mathrm{G}/K_{\infty}$, where

$$K_{\infty} = \mathrm{Stab}_{\mathrm{G}(\mathbb{R})}(h)$$

is a maximal compact subgroup of G($\mathbb{R}$) after modulo Z(G).

(2) The Condition ($SV3$) can be removed in the definition.

We will first see some examples and then come back to talk about each condition in detail.

Example 1.6. Let $\mathrm{G} = \mathrm{GL}_2$ over $\mathbb{Q}$. Then $\mathrm{G}^{\mathrm{ad}} = \mathrm{PGL}_2$. Define $h : \mathbb{S} \to \mathrm{GL}_2$ such that

$$h(x+iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

By Example 1.2, $\mathrm{ad}(h(i))$ is a Cartan involution on G^{ad}.

Since $h(\mathbb{S})$ is a commutative subgroup of $\mathrm{GL}_2(\mathbb{C})$, there exists an element $g \in \mathrm{GL}_2(\mathbb{C})$ such that

$$gh(re^{it})g^{-1} = diag(re^{it}, re^{-it})$$

for every $z = re^{it} \in \mathbb{C}^{\times}$. Therefore, the eigenvalues of $\mathrm{ad}(h(z))$ acting on $\mathrm{gl}_2(\mathbb{C})$ are $e^{2it} = \frac{z}{\overline{z}}, 1, 1$ and $e^{-2it} = \frac{\overline{z}}{z}$.

The stabilizer of h is

$$\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{R})}(h) = \{g\mathrm{GL}_2(\mathbb{R}) \mid g\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)g^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\} = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)\} \simeq \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^{\times}.$$

Thus, $X \simeq \mathrm{GL}_2(\mathbb{R})/(\mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^{\times}) \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \times \{\pm 1\} \simeq \mathfrak{H}^{\pm}$ by mapping $\mathrm{Ad}_g(h)$ to $\frac{ai+b}{ci+d}$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Example 1.7. Let $G = GSp_{2g}$ over $\mathbb{Q}$ with respect to the sympletic form $\left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. Define $h : \mathbb{S} \to G$ such that

$$h(x+iy) = \left(\begin{smallmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{smallmatrix}\right).$$

Example 1.8. Let D be a quaternion algebra over a totally real field F. Put

$$\Sigma_\infty := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R}) = \{\tau_1, \tau_2, \cdots, \tau_d\},$$

$$\Sigma_\infty^{\mathrm{ur}} = \{\tau \in \Sigma_\infty \mid \mathrm{D} \text{ is unramified at } \tau\},$$

$$\Sigma_\infty^{\mathrm{ram}} = \{\tau \in \Sigma_\infty \mid \mathrm{D} \text{ is ramified at } \tau\}.$$

Then we have

$$\mathrm{D} \otimes_{F,\tau} \mathbb{R} \simeq \begin{cases} \mathrm{M}_2(\mathbb{R}), & \tau \in \Sigma_\infty^{\mathrm{ur}} \\ \mathbb{H}, & \tau \in \Sigma_\infty^{\mathrm{ram}} \end{cases}$$

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{D}^\times)$. Then

$$G_{\mathbb{R}} \simeq \prod_{\tau \in \Sigma_\infty^{\mathrm{ur}}} \mathrm{GL}_{2,\mathbb{R}} \times \prod_{\tau \in \Sigma_\infty^{\mathrm{ram}}} \mathbb{H}^\times$$

because $\mathrm{D} \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\tau \in \Sigma_\infty} D \otimes_{F,\tau} \mathbb{R}$. Define $h : \mathbb{S} \to G_{\mathbb{R}}$ such that

$$h(x+iy) = ((\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right))_{\tau \in \Sigma_\infty^{\mathrm{ur}}}, (1)_{\tau \in \Sigma_\infty^{\mathrm{ram}}})$$

Since $\mathbb{H} = \{\left(\begin{smallmatrix} a & -b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})\}$, we see $\mathbb{H}^{\times,\mathrm{ad}} = \mathrm{SU}_2$. By using Fact 1.4, we see that $h(i)$ is a Cartan involution on G^{ad}.

By Example 1.6, we get $X \simeq (\mathfrak{H}^\pm)^{\Sigma_\infty^{\mathrm{ur}}} \times \{1\}$.

Example 1.9. Let $E/\mathbb{Q}$ be an imaginary quadratic extension and V be a Hermitian space of dimension n over E with signature (a, b). The group $G = GU(V)$. For a $\mathbb{Q}$-algebra R,

$$G(R) = \{(g, c) \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \times R^\times \mid < gx, gy >= c < x, y > \ \forall x, y \in V \otimes_{\mathbb{Q}} R\}.$$

The homomorphism $h : \mathbb{S}(\mathbb{R}) \to G(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C})$ maps $z \in \mathbb{C}^\times$ to the diagonal matrix $\mathrm{diag}(z, z, \ldots, z, \bar{z}, \bar{z}, \ldots, \bar{z})$ with respect to the Hermitian form

$$\mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1).$$

Then $\mathrm{Stab}_{G(\mathbb{R})}(h) = G(U(a) \times U(b))$ and $X = G(\mathbb{R})/\mathrm{Stab}_{G(\mathbb{R})}(h)$. The dimension of X is $\dim_{\mathbb{C}} X = ab$.

A typical h for the unitary group is $h' : \mathbb{S}(\mathbb{R}) \to U(V)(\mathbb{R})$ such that

$$h'(z) = \mathrm{diag}(1, 1, \ldots, 1, \frac{\bar{z}}{z}, \frac{\bar{z}}{z}, \ldots, \frac{\bar{z}}{z}).$$

Remark 1.2. Somehow, h and h' are not quite the same. e.g. h is not the composition $\mathbb{S} \xrightarrow{h'} \mathrm{U}(V)_{\mathbb{R}} \hookrightarrow \mathrm{GU}(V)_{\mathbb{R}}$. So going from $\mathrm{GU}(V)$ to $\mathrm{U}(V)$ is a bit tricky.

Example 1.10. Assume G is a reductive group over $\mathbb{Q}$ such that $\mathrm{G}^{\mathrm{ad}}(\mathbb{R})$ is compact. Define $h : \mathbb{S} \to \mathrm{G}_{\mathbb{R}}$ such that $h(z) = 1$. This satisfies $(SV1), (SV2)$ but not $(SV3)$.

1.2. **Hogde Structure.** We shall explain conditions in the Definition1.5, especially how it is related to the variation of Hodge structures.

Definition 1.11. (See [Del2, Section 2.1]) A *Hodge structure* on a vecter space V over $\mathbb{Q}$ is the following equivalent structure:

(1) a homomorphism $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$;

(2) a bigrading decomposition $V_{\mathbb{C}} = \oplus_{i,j} V^{i,j}$ such that $\overline{V^{i,j}} = V^{j,i}$. (Convention is that $h(z)$ acts on $V^{i,j}$ as the multiplication by $z^{-i}\overline{z}^{-j}$.)

We say the Hodge structure on V is *pure of weight* n, if the following equivalent condition holds:

(1) $h_{|\mathbb{G}_m}$ is $x \mapsto x^{-n}$.

(2) $V^{i,j} = 0$ unless $i + j = n$.

The *Hodge filtration* on $V_{\mathbb{C}}$ is given by $\mathrm{Fil}^i V_{\mathbb{C}} = \sum_{k \geq i} V^{k,j}$.

Example 1.12. ([Del2, Définition 2.1.13]) The *Hodge structure of Tate* $\mathbb{Q}(1)$ is a Hodge structure on $V = 2\pi i\mathbb{Q} \subset \mathbb{C}$ which is pure of weight -2 and of the Hodge type $(-1, -1)$ (i.e. $V_{\mathbb{C}} = V^{-1,-1}$). ($\Leftrightarrow h : \mathbb{S} \to \mathrm{GL}(2\pi i\mathbb{R}), z \mapsto |z|^2$).

Definition 1.13. ([Del2, Définition 2.1.15]) Endow V with a pure Hodge structure ($\Leftrightarrow h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$) of weight n. A *polarization* is a homomorphism of Hodge structures $< -, - >: V \otimes V \to \mathbb{Q}(-n)$ such that the bilinear form on $V_{\mathbb{R}}$

$$x \otimes y \mapsto (2\pi i)^n < x, h(i)y >$$

is symmetric and positive definite.

Remark 1.3. By definition, for any $x, y \in V_{\mathbb{R}}$, we have

$$< x, y >=< h(i)x, h(i)y >=< y, h(i)^2 x >=< y, h(-1)x >= (-1)^n < y, x > .$$

Thus, the bilinear form $< -, - >$ is symmetric when n is even and is alternative when n is odd.

Definition 1.14. Let S be a complex analytic manifold. A *variation of Hodge structure* consists of

(1) a $\mathbb{Q}$-local system $\underline{V}$ on S and

(2) a decreasing filtration $\mathrm{Fil}^p\mathcal{V}$ of $\mathcal{V} = \underline{V}\otimes\mathcal{O}_S$ satisfying Griffiths transversality, i.e. $\nabla(\mathrm{Fil}^p\mathcal{V}) \subset \mathrm{Fil}^{p-1}\mathcal{V}\otimes\Omega^1_S$, such that for every $s \in S$, this filtration gives a Hodge structure on $\underline{V}_s$.

Theorem 1.15. *Let (G, X) be a Shimura datum and let $\mathrm{G} \hookrightarrow \mathrm{GL}(V)$ be a faithful $\mathbb{Q}$-representation of* G

(1) *The part in (SV_1) claiming that the Hodge structure induced by $\mathbb{S} \xrightarrow{\mathrm{Ad}\circ h} \mathrm{GL}(\mathrm{Lie}(\mathrm{G})_{\mathbb{R}})$ is pure of weight 0 implies that there exists a unique complex structure on X such that the Hodge filtration on $\underline{V}_X$ induced by $\mathbb{S} \xrightarrow{h} \mathrm{G}_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$ varies holomorphically.*

(2) *Under (1), (SV_1)(meaning the Hodge structure on $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$ is of types $(1,-1),(0,0),(-1,1)$) is equivalent to that the filtration for above Hogde structure satisfies the Griffiths transversallity.*

(3) *When V is pure of weight n, (SV_2) implies that there is a polarization*

$$< -, - >: \underline{V}_X \times \underline{V}_X \to \underline{\mathbb{R}}_X(-n).$$

Proof. (1) Here we only prove the existence; working more carefully gives the uniqueness. Since the Hodge structure on $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$ is pure of weight 0, the $h_{|\mathbb{G}_m}$ acts trivially on $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$. Thus $h(\mathbb{G}_m) \subset \mathrm{Z}(\mathrm{G}_{\mathbb{R}})^{\circ}$. We may assume V is irreducible. Then $\mathbb{G}_m \xrightarrow{h} \mathrm{Z}(\mathrm{G}_{\mathbb{R}}) \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$ acts by $x \mapsto x^{-n}$ for some $n \in \mathbb{Z}$.

Each $h \in X$ defines a Hodge structure on $V =: V_h$ (induced by $\mathbb{S} \xrightarrow{h} \mathrm{G}_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$). More precisely, there exists a decomposition (of eigenspaces of h)

$$V_{h,\mathbb{C}} = \oplus_{i+j=n} V_h^{i,j}$$

such that $h(z)$ acts on $V_h^{i,j}$ as multiplication by $z^{-i}\overline{z}^{-j}$. The Hodge filtration is given by $\mathrm{Fil}^i V_{h,\mathbb{C}} = \sum_{k\geq i} V_h^{k,j}$. Then $V^{i,j} = \mathrm{Fil}^i V_{h,\mathbb{C}} \cap \overline{\mathrm{Fil}^j V_{h,\mathbb{C}}}$. Since any element in X is a $\mathrm{G}(\mathbb{R})$-conjugate of h, $\dim \mathrm{Fil}^i V_{h,\mathbb{C}}$ is independent of h.

Then we get a morphism

$$\phi : X \to \mathcal{F}(V_{\mathbb{C}}, (\dim \mathrm{Fil}^i V_{h,\mathbb{C}})_i), \quad h \mapsto (\mathrm{Fil}^i V_{\mathbb{C}})$$

where $\mathcal{F}(V_{\mathbb{C}}, (\dim \mathrm{Fil}^i V_{h,\mathbb{C}})_i)$ is a flag variety.

Now, we verify that X is an analytic sub-variety. Considering the tangent morphism $d\phi_h$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{T}_h X = \mathrm{Lie}(\mathrm{G})_{\mathbb{R}}/\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}^{(0,0)} & \longrightarrow & \mathrm{End}(V_{\mathbb{R}})/\mathrm{End}(V_{\mathbb{R}})^{(0,0)} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Lie}(\mathrm{G})_{\mathbb{C}}/\mathrm{Fil}^0\mathrm{Lie}(\mathrm{G})_{\mathbb{C}} & \xrightarrow{\text{inj.}} & \mathrm{End}(V_{\mathbb{C}})/\mathrm{Fil}^0\mathrm{End}(V_{\mathbb{C}}) \quad = \mathrm{T}_{\phi(h)}\mathcal{F}(V_{\mathbb{C}}, (\dim \mathrm{Fil}^i V_{h,\mathbb{C}})_i) \end{array}.$$

We explain contents here.

(•1) Since V admits a pure Hodge structure of weight n, $\mathrm{End}(V) \simeq V \otimes V^*$ admits a pure Hodge structure of weight 0. In this situation $\mathrm{Fil}^0\mathrm{End}(V)_{\mathbb{C}}$ consists of all endomorphisms which preserve Hodge filtration of V (See [Del2, Section 1]).

(•2) If V admits a pure Hodge structure of weight 0, then the morphism

$$V_{\mathbb{R}}/V_{\mathbb{R}}^{(0,0)} \to V_{\mathbb{C}}/\mathrm{Fil}^0 V_{\mathbb{C}}$$

induced by $V_{\mathbb{R}} \to V_{\mathbb{C}}$ is an isomorphism, where $V_{\mathbb{R}}^{(0,0)} = V_{\mathbb{R}} \cap V^{0,0}$. (by counting dimension).

(•3) Since $X = \mathrm{G}(\mathbb{R})/\mathrm{Stab}_{\mathrm{G}(\mathbb{R})}(h)$, $\mathrm{T}_h X$ is a quotient of $\mathrm{Lie}(\mathrm{G})_{\mathbb{R}}$ by the subspace on which $\mathbb{S}$ acts trivially. This subspace is contained in $\mathrm{Lie}(\mathrm{G})_{\mathbb{C}}^{0,0}$ because $\mathbb{S}$ acts on $V^{i,j}$ as $z^{-i}\overline{z}^{-j}$.

(•4) $\mathcal{F}(V_{\mathbb{C}}, (\dim \mathrm{Fil}^i V_{h,\mathbb{C}})_i)$ is a flag variety and hence is isomorphic to GL/P for the parabolic group determined by $(\dim \mathrm{Fil}^i V_{h,\mathbb{C}})_i$. Thus, its tangent space consists of all elements in $\mathrm{End}(V_{\mathbb{C}})$ which preserve the filtration.

This defines a natural complex structure on X.

(2) The Griffiths transversality translate to that the image of $d\phi$ lies in $\mathrm{Fil}^{-1}\mathrm{End}(V_{\mathbb{C}})/\mathrm{Fil}^0(\mathrm{End}(V_{\mathbb{C}}))$ which is equivalent to that Hodge types of Lie(G) can only be $(1,-1), (0,0), (-1,1)$. (Becauce $\mathrm{Lie}(\mathrm{G})_{\mathbb{C}}/\mathrm{Fil}^0\mathrm{Lie}(\mathrm{G})_{\mathbb{C}} \subset \mathrm{Fil}^{-1}\mathrm{End}(V_{\mathbb{C}})/\mathrm{Fil}^0\mathrm{End}(V_{\mathbb{C}})$)

(3) This is some result from Lie theory and we omit here.

□

2. Classification of Shimura data

Recall that $\mathbb{S}_{\mathbb{C}} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \times_{\mathrm{Spec}\mathbb{R}} \mathrm{Spec}\mathbb{C} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. For $\mathbb{C}$-points, the isomorphism $\mathbb{S}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C} \otimes \mathbb{C}) \xrightarrow{\simeq} \mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$ is given by $a \otimes b \mapsto (ab, a\bar{b})$. Thus for any $h : \mathbb{S} \to \mathrm{G}$, it induces an $h_{\mathbb{C}} : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to \mathrm{G}_{\mathbb{C}}$.

Define $\mu : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} \mathrm{G}_{\mathbb{C}}$ (G($\mathbb{C}$)-conjugate class) Hodge cocharacter attached to Shimura datum (G, X)

Fact 2.1. *There is an one-to-one correspondence between the set*

$$\{\mathrm{G}^{\mathrm{ad}}(\mathbb{R}) - \text{conjugate classes of } h : \mathbb{S} \to \mathrm{G}_{\mathbb{R}} \text{satisfying } (SV_1)\text{and}(SV_2)\}$$

and the set

$$\{\mathrm{G}(\mathbb{C}) - \text{conjugate classes of } \mu : \mathbb{G}_m \to \mathbb{G}_{\mathbb{C}} \text{ that are minuscule.}\}$$

When G *is simple adjoint, this is a bijection.*

One can find a proof (for simple adjoint groups) in [Del1, Proposition 1.2.2].

Now we explain the meaning of "minuscule".

For a cocharacter $\mu : \mathbb{G}_m \to \mathrm{G}$, by choosing a suitable base of root system Δ (or a suitable maximal torus T), we may assume $\mu \in \mathrm{X}_*(\mathrm{T})^{\mathrm{dom}}$; that is, all coefficients of the expansion of μ with respect to this base are non-negative. Choose Langland dual group $\hat{\mathrm{G}}$ (i.e. reductive group with dual root system $\hat{\Delta}$) such that

$$(\mathrm{X}^*(\hat{\mathrm{T}}), \hat{\Delta}, \mathrm{X}_*(\hat{\mathrm{T}}), \hat{\Delta}^\vee) = (\mathrm{X}_*(\mathrm{T}), \Delta, \mathrm{X}^*(\mathrm{T}), \Delta^\vee)$$

Then $\mu \in \mathrm{X}^*(\hat{\mathrm{T}})^{\mathrm{dom}}$ and hence there is a irreducible representation (highest weight representation)

$$\rho_\mu : \hat{G} \to \mathrm{GL}(V_\mu)$$

with highest weight μ (See [Hum, Theorem 31.4]). We say μ is *minuscule* if all weights in V_μ have multiplicity 1.

When $\mathrm{G}_{\mathbb{R}}$ is simple adjoint, μ is minuscule if and only if in the action of $\mathbb{G}_m$ on $\mathrm{Lie}(\mathrm{G}_{\mathbb{C}})$ defined by $\mathrm{ad} \circ \mu$, only the characters $z, 1, z^{-1}$ occur.

Before classifying Shimura data for simple groups, we give the following definitions.

Definition 2.2. Let (G, X) be a Shimura datum.

(1) We say (G, X) is of *Hodge type* if there is an embedding $\mathrm{G} \hookrightarrow \mathrm{GSp}_{2g}$ such that the composition $\mathbb{S} \xrightarrow{h} \mathrm{G}_{\mathbb{R}} \hookrightarrow \mathrm{GSp}_{2g,\mathbb{R}}$ is conjugate to

$$z = x + iy \mapsto \left(\begin{smallmatrix} xI_g & -yI_g \\ yI_g & xI_g \end{smallmatrix}\right)$$

(See Example 1.7.)

(2) We say (G, X) is of *abelian type* if there is a Shimura datum (G', X') of Hodge type such that there is an isogeny $\mathrm{G}'^{,\mathrm{der}} \to \mathrm{G}^{\mathrm{der}}$ inducing an isomorphism

$$\mathrm{G}'^{,\mathrm{ad}}(\mathbb{R})X' \xrightarrow{\simeq} \mathrm{G}^{\mathrm{ad}}(\mathbb{R})X.$$

Now, we classify Shimura datum for "essentially simple" groups. We only need to classify $\hat{\mathrm{G}}$ and minuscule cocharacter μ by Fact 2.1. We just list the result here.

Type:= Type of G.

$\mathbf{TypeA}$: $\hat{\mathrm{G}} = \mathrm{GL}_n, \mu = (1, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}^n \simeq \mathrm{X}_*(\mathrm{T}_{\mathbb{C}})$. Then ρ_μ : $\hat{\mathrm{G}} \xrightarrow{\wedge^a} \mathrm{GL}_{\binom{n}{a}}(\mathbb{C})$. In this case, $\mathrm{G} = \mathrm{U}(a, n-a)$ and $h : \mathbb{S} \to \mathrm{G}_{\mathbb{R}}$ maps z to $diag(\frac{z}{\bar{z}}, \cdots, \frac{z}{\bar{z}}, 1, \cdots, 1)$. The Shimura data is of abelian type.

$\mathbf{TypeB}$: $\hat{\mathrm{G}} = \mathrm{GSp}_{2n}$, μ is the vector representation. Then

$$\mathrm{G} = \mathrm{GSpin}(2, 2n-1) \hookrightarrow \mathrm{GSp}_{2^{2n}}$$

and the Shimura data is of abelian type.

$\mathbf{TypeC, D}$: $\hat{\mathrm{G}} = \mathrm{GSpin}_n$ and μ is the Spin representation. Then

$$\mathrm{G} = \begin{cases} \mathrm{GSp}_{n-1}, & n \text{ is odd} \\ \mathrm{GSpin}(2, n-2), & n \text{ is even} \end{cases}$$

Shimura data are both of abelian type.

$\mathbf{TypeD^H}$: $\hat{G} = \mathrm{GSO}_{2n}$ and μ is the vector representation. Then

$$\mathrm{G} = \mathrm{GSO}_{2n} \hookrightarrow \mathrm{GSp}_{4n}.$$

and the Shimura data is of Hodge type.

$\mathbf{E_6, E_7}$: There are minuscule representations but not of abelian type.

Remark 2.1. (1) Type A, Type C and Type D^H are of "PEL" type which we shall discuss in later.

(2) Here we explain how to give the classification by using Dynkin diagrams. For a cocharacter μ, we choose a suitable base $\{\alpha_i\}_{i \in I}$ of the root system Δ of G such that $\mu \in \mathrm{X}_*(\mathrm{T})$. Then μ is minuscule if and only if $< \mu, \alpha > \in \{\pm 1, 0\}$ for all roots α. Let $\tilde{\alpha} = \sum_{iI} n_i \alpha_i$ be the highest root. The condition that $< \mu, \tilde{\alpha} > \in \{\pm 1, 0\}$ implies that $\mu = \alpha_i$ for some $i \in I$ satisfying $n_i = 1$ (because $\mu \in \mathrm{X}_*(\mathrm{T})^{\mathrm{dom}}$). Such an α_i is called a special node. Then the above classification follows from the classification of special nodes on Dynkin diagrams. (See [Mil, Theorem 1.25])

3. Shimura Reciprocity Law and Canonical Model

We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Put $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Let (G, X) be a Shimura datum, then there is a $\mathrm{G}(\mathbb{C})$-conjugate class of Hogde cocharacter $\{\mu : \mathbb{G}_m \to \mathrm{G}_{\mathbb{C}}\}$. This is a natural variety and can be defined over a number field $E \subset \bar{\mathbb{Q}}$. This is called the *reflex field*. Explicitly, we may assume $\mu \in \mathrm{X}_*(\mathrm{T}_{\mathbb{C}})^{\mathrm{dom}}$, on which $G_{\mathbb{Q}}$ acts naturally. Then

$$E := E(\mathrm{G}, X) = \text{subfield of } \bar{\mathbb{Q}} \text{ fixed by } \mathrm{Stab}_{G_{\mathbb{Q}}}(\mu).$$

Now, we give two examples.

Example 3.1. Let $E/\mathbb{Q}$ be a imaginary quadratic extension with Galois group $< \sigma >$ and let V be an Hermitian space over E of dimension n and signature $(a, n-a)$. The homomorphism $h : \mathbb{S} \to \mathrm{GU}(V)$ is given by $z \mapsto diag(z, \cdots, z, \overline{z}, \cdots, \overline{z})$. (See Example1.9)

Then $\mathrm{GU}(V)_{\mathbb{C}} \simeq \mathrm{GL}_{n,\mathbb{C}} \times \mathbb{G}_m$ and $\simeq \mathbb{Z}^n \oplus \mathbb{Z}$. The action of $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ on $\mathrm{X}_*(\mathrm{GL}_n \times \mathbb{G}_m) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$ is given by

$$(a_1, a_2, \dots, a_n; b) \mapsto (b - a_1, b - a_2, \dots, b - a_n; b).$$

The cocharacter associated to h is $\mu_h = (1, \dots, 1, 0, \dots, 0; 1) \in \mathrm{X}_*(\mathrm{GL}_n \times \mathbb{G}_m)$.

Thus, if $a = n - a$, then μ is invariant under σ-action (up to a $\mathrm{GU}(V)(\mathbb{C})$-conjugation) and the reflex field is $\mathbb{Q}$ in this case; if $a \neq n - a$, then the reflex field is E.

Example 3.2. Let F be a total real field and $\mathrm{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$. Then $\mathrm{G}_{\mathbb{R}} = \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_{2,\mathbb{R}}$ (See Example 1.8). The homomorphism $h : \mathbb{S} \to \mathrm{G}_{\mathbb{R}}$ is given by

$$z \mapsto (\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right))_{\tau \in \Sigma_\infty}.$$

The cocharacter associated to h is $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_{2,\mathbb{C}}$ mapping z to $(diag(z, 1))_\tau$.

Now, $\mathrm{X}_*(\mathrm{G}_{\mathbb{C}}) = \prod_{\tau \in \Sigma_\infty} \mathbb{Z}^2$ with $G_{\mathbb{Q}}$ actting on the right hand side as permutations on Σ_∞. We note that $\mu_h = ((1, 0))_\tau \in \prod_{\tau \in \Sigma_\infty} \mathbb{Z}^2$. Therefore, μ_h is invariant under Galois actions (up to $\mathrm{G}(\mathbb{C})$-conjugations). It follows that the reflex field is $\mathbb{Q}$.

Theorem 3.3. *The tower of Shimura variety* $\mathrm{Sh}(\mathrm{G}, X) = (\mathrm{Sh}_K(G, X))_{K \subset \mathrm{G}(\mathbb{A}_f)}$ *admits a* canonical model *over the reflex field E. Here, the* $\mathbb{C}$*-points of* $\mathrm{Sh}_K(\mathrm{G}, X)$ *is given by*

$$\mathrm{Sh}_K(\mathrm{G}, X)(\mathbb{C}) = \mathrm{G}(\mathbb{Q}) \setminus X \times (\mathrm{G}(\mathbb{A}_f)/K)$$

Now, we explain the meaning of "canonical model".

We give a typical example at first.

Assume $\mathrm{G} = \mathrm{T}^d$ is a torus of dimension d over $\mathbb{Q}$; that is $\mathrm{T}^d_{\mathbb{C}} \simeq \mathbb{G}^d_{m,\mathbb{C}}$. In this case, $\mathrm{X} = \{h\}$ consists exactly one element $h : \mathbb{S} \to \mathrm{T}^d_{\mathbb{R}}$ and then $\mathrm{Sh}_K(\mathrm{T}^d)(\mathbb{C}) = \mathrm{T}^d(\mathbb{Q}) \setminus \mathrm{T}^d(\mathbb{A}_f)/K$ for any open compact subgroup $K \subset \mathrm{T}^d(\mathbb{A}_f)$ is a finite set. The cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{T}^d_{\mathbb{C}}$ is defined over the reflex field E; that is, μ_h is induced by some $\mu : \mathbb{G}_{m,E} \to \mathrm{T}^d_E$. Then we get a homomorphism of groups $\mathrm{Res}_{E/\mathbb{Q}}(\mu) : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to \mathrm{Res}_{E/\mathbb{Q}}\mathrm{T}^d_E$. We denote by $NR_E(\mu)$ the composition

$$\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}(\mu)} \mathrm{Res}_{E/\mathbb{Q}}\mathrm{T}^d_E \xrightarrow{N_{E/\mathbb{Q}}} \mathrm{T}^d.$$

For any extension F/E (μ is obviously defined over F), we see $NR_F(\mu) = NR_E(\mu) \circ N_{F/E}$.

$$\begin{array}{ccc} \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F} & & \\ & \searrow^{NR_F(\mu)} & \\ \downarrow{\scriptstyle N_{F/E}} & & \mathrm{T}^d \\ & \nearrow_{NR_E(\mu)} & \\ \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} & & \end{array}.$$

This induces a morphism

$$E^\times \setminus \mathbb{A}_E^\times / E_{\mathbb{R}}^{\times,\circ} \xrightarrow{NR(\mu)} \mathrm{T}^d(\mathbb{Q}) \setminus \mathrm{T}^d(\mathbb{A}_f).$$

and we get the *Shimura reciprocity map* $\mathrm{Res}_\mu : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathrm{T}^d(\mathbb{Q}) \setminus \mathrm{T}^d(\mathbb{A}_f)$

$$\mathrm{Gal}(\bar{\mathbb{Q}}/E) \xrightarrow{\text{surj.}} \mathrm{Gal}(E^{ab}/E) \xleftarrow{\text{Art}} E^\times \setminus \mathbb{A}_E^\times / E_{\mathbb{R}}^{\times,\circ} \xrightarrow{NR(\mu)} \mathrm{T}^d(\mathbb{Q}) \setminus \mathrm{T}^d(\mathbb{A}_f),$$

where the morphism Art is the Artin reciprocity map which takes uniformizer to the geometric Frobenius.

Let $K \subset \mathrm{T}^d(\mathbb{A}_f)$. The *canonical model* of $\mathrm{Sh}_K(\mathrm{T}^d)$ is an E-scheme $\mathrm{M}_K(\mathrm{T}^d)$ satisfying $\mathrm{M}_K(\mathrm{T}^d)(\mathbb{C}) = \mathrm{Sh}_K(\mathrm{T}^d)$ such that the action of $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ on $\mathrm{Sh}_K(\mathrm{T}^d)$ is given by the right-translation by $\mathrm{Res}_\mu(\tau)$, i.e. for $x \in \mathrm{M}_K(\mathrm{T}^d)(\mathbb{C}) = \mathrm{T}^d(\mathbb{Q}) \setminus \mathrm{T}^d(\mathbb{A}_f)/K$ and $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$, $\tau(x) = x \cdot \mathrm{Res}_\mu(\tau)$.

Now, we come back to the general case. For a Shimura datum (G, X) a *canonical model* of the Shimura variety $\mathrm{Sh}_K(\mathrm{G}, X)$ is an $E(= E(\mathrm{G}, X))$-scheme $\mathrm{M}_K(\mathrm{G}, X)$ satisfying $\mathrm{M}_K(\mathrm{G}, X)(\mathbb{C}) = \mathrm{Sh}_K(\mathrm{G}, X)$ such that for any morphism of Shimura data $(\mathrm{T}^d, \{h\}) \to (\mathrm{G}, X)$, the natural morphism

$$\mathrm{T}^d(\mathbb{Q}) \setminus \{h\} \times \mathrm{T}^d(\mathbb{A}_f)/K \cap \mathrm{T}^d(\mathbb{A}_f) \to \mathrm{G}(\mathbb{Q}) \setminus X \times (\mathrm{G}(\mathbb{A}_f)/K)$$

is induced by a morphism $\mathrm{M}_{K \cap \mathrm{T}^d(\mathbb{A}_f)}(\mathrm{T}^d, \{h\}) \to \mathrm{M}_K(\mathrm{G}, X) \times_E E(\mathrm{T}^d, \{h\})$.

Finally, we remark that there are "enough" such morphisms

$$(\mathrm{T}^d, \{h\}) \to (\mathrm{G}, X)$$

to rigidify the scheme structure of $\mathrm{M}_K(\mathrm{G}, T^d)$.

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