General theory of Shimura varieties

Ultimate goal: Explain the meaning of a writing $\mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ we've seen many times.

§1. Shimura data

Definition. A Cartan involution θ on a linear algebraic group G over $\mathbb{R}$ is an automorphism θ of G over $\mathbb{R}$ s.t.

(1) $\theta^2 = id$ and

complex conjugation w.r.t. the $\mathbb{R}$-structure of G

(2) $G^{(\theta)}(\mathbb{R}) := \{ g \in G(\mathbb{C}) \mid \theta(g) = \bar{g} \}$ is compact.

Blackbox Theorem G has a Cartan involution if and only if G is reductive.

In this case, two Cartan involutions are differed by conjugation by an elt of $G(\mathbb{R})$

We now first give the definition of Shimura data & then discuss what it entails.

Definition A Shimura datum consists of a pair (G,X).

- G is a reductive group $/\mathbb{Q}$
- X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h: \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$

s.t. for one (and thus every) h, the following condition holds

(SV1) the composition $\mathbb{S} \to G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathfrak{g}_{\mathbb{R}}$ defines a Hodge structure of type $(-1,1), (0,0), (1,-1)$. (i.e. $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{g}_{\mathbb{R}}$ with eigenvalues $\frac{z}{\bar{z}}$, 1, $\frac{\bar{z}}{z}$.)

(SV2) Conjugation by the image of $h(i)$ in $G^{ad} = G/\text{center of } G$ is a Cartan involution.

(SV3) For every $\mathbb{Q}$-simple factor H of G^{ad}, $H(\mathbb{R})$ is not compact. ← can be removed as well.

There are other "axioms" that simplify situations/discussions.

The question is which generality we will allow.

Remark: By definition, $X \cong G/K_\infty$ with K_∞ = stabilizer of $h(\mathbb{S})$

in practice $\cong$ stabilizer of $h(i) \overset{(SV2)}{=\!=} $ a max'l compact (mod center) subgp of $G(\mathbb{R})$

Example: $G = \mathrm{Gl}_2/\mathbb{Q}$ $\quad h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow \mathrm{Gl}_2(\mathbb{R})$

$$z = x+iy \longmapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

The action of $\mathrm{Ad}\circ h_0(x)$ acts trivially on $\mathfrak{gl}_2$, $\mathrm{Ad}\circ h_0(yi)$ acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, eigenvalues are 2 copies of 1 & -1

$$\mathrm{Stab}_{h_0}(\mathrm{Gl}_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} = \underbrace{O(2)\cdot\mathbb{R}^\times}_{\text{compact mod center}}$$

$\Rightarrow$ (SV2)

$$X = \mathrm{Ad}_{\mathrm{Gl}_2(\mathbb{R})}(h_0) \longrightarrow \mathfrak{h}^\pm$$

$$\mathrm{Ad}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(h_0) \longmapsto \frac{ai+b}{ci+d}$$

Example: $G = \mathrm{GSp}_{2g}/\mathbb{Q}$ $\quad$ w.r.t. symplectic form $\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$

$$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow \mathrm{GSp}_{2g}(\mathbb{R})$$

$$x+iy \longmapsto \begin{pmatrix} xI_g & -yI_g \\ yI_g & xI_g \end{pmatrix}$$

Example: D quaternion alg$/F$, with F totally real. $\mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{R}) = \{\tau_1, \cdots, \tau_d\}$

$$D \otimes_{F, \tau_i} \mathbb{R} = \begin{cases} M_2(\mathbb{R}) & \text{if } D \text{ is unramified at } \tau_i \quad \leftarrow \Sigma_\infty^{\mathrm{unr}} \ni \tau_i \\ \mathbb{H} & \text{if } D \text{ is ramified at } \tau_i \quad \leftarrow \Sigma_\infty^{\mathrm{ram}} \ni \tau_i \end{cases}$$

$G = \mathrm{Res}_{F/\mathbb{Q}}(D^\times)$,

$$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G_{\mathbb{R}} \cong \prod_{\tau\in\Sigma_\infty^{\mathrm{unr}}} \mathrm{Gl}_{2,\mathbb{R}} \times \prod_{\tau\in\Sigma_\infty^{\mathrm{ram}}} \mathbb{H}^\times$$

$$z = x+iy \longmapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix} , \; 1 \right)$$

$\uparrow$ usually uses 1, but can also use x^2+y^2 or $(x^2+y^2)^{-1}$ depending on the needs.

$$\rightsquigarrow X = \mathrm{Ad}_{G(\mathbb{R})} h_0 \cong (\mathfrak{h}^\pm)^{\Sigma_\infty^{\mathrm{unr}}} \times \{1\}.$$

When $\Sigma_\infty^{\mathrm{ram}} \neq \phi$, the map h_0 factors through $G' = (\mathrm{Res}_{F/\mathbb{Q}} D^\times)^{\det\in\mathbb{Q}^\times}$

$$G'_{\mathbb{R}} = \Big(\prod_{\tau\in\Sigma_\infty} \mathrm{Gl}_{2,\mathbb{R}}\Big)^{\det\ \underline{\underline{\text{same}}}}$$

$$h_0' : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G'_{\mathbb{R}} \cong \Big(\prod_{\tau\in\Sigma_\infty} \mathrm{Gl}_{2,\mathbb{R}}\Big)^{\det\ \text{same}}$$

$$z = x+iy \longmapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right)$$

$$X' := \mathrm{Ad}_{G'_{\mathbb{R}}}(h_0') = (\mathfrak{h}^+)^{\Sigma_\infty^{\mathrm{unr}}} \sqcup (\mathfrak{h}^-)^{\Sigma_\infty^{\mathrm{unr}}}$$

<u>Example</u>: $\begin{matrix} E \\ | \\ \mathbb{Q} \end{matrix}$ imaginary quadratic field. V a Hermitian space of dim n/E. signature (a,b) $a+b=n$

The group $G = GU(V)$, for a $\mathbb{Q}$-algebra R,

$$G(R) = \{(g,c) \in GL(V \otimes_{\mathbb{Q}} R) \times R^\times \mid \langle gx, gy\rangle = c\langle x,y\rangle \ \forall x,y \in V \otimes_{\mathbb{Q}} R\}$$

$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G(\mathbb{R}) \subseteq GL_n(\mathbb{C})$ ↖ w.r.t. Hermitian matrix

$$z \longmapsto \begin{pmatrix} \overbrace{z \ \ddots \ z}^{a} & \\ & \overbrace{\bar z \ \ddots \ \bar z}^{b} \end{pmatrix} \qquad \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{a} & \\ & \overbrace{-1 \ \ddots \ -1}^{b} \end{pmatrix}$$

$$X = \mathrm{Ad}_{G(\mathbb{R})}(h_0) \qquad \mathrm{Stab}_{h_0}(G(\mathbb{R})) = G(U(a) \times U(b))$$

A typical h_0 for unitary group is $h_0' : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow U(V)(\mathbb{R})$

$$z \longmapsto \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{a} & \\ & \overbrace{z/\bar z \ \ddots \ z/\bar z}^{b} \end{pmatrix}$$

<u>Remark</u>: Somehow, h_0 & h_0' are not quite the same. e.g. h_0 is not $\mathbb{S} \xrightarrow{h'} U(V)_{\mathbb{R}} \to GU(V)_{\mathbb{R}}$

So going from $\underbrace{GU(V)}_{\text{better moduli problem}}$ to $\underbrace{U(V)}_{\text{better automorphic stories.}}$ is a bit tricky

Will explain more about this later

* We now explain the conditions in the definition, especially how it's related to variation of Hodge structure

<u>Recall</u>: A <u>Hodge structure</u> on a $\mathbb{Q}$-vector space V is the following equivalent structure:

① a homomorphism $h : \mathbb{S} \to GL_{\mathbb{R}}(V_{\mathbb{R}})$

② a bigrading decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} V^{pq}$ s.t. $\overline{V^{pq}} = V^{qp}$

We say V has pure <u>weight n</u> if the following equivalent conditions hold:

① $h|_{\mathbb{G}_m}$ is $x \longmapsto x^{-n}$

② $V^{pq} = 0$ unless $n = p+q$.

Let S be a complex analytic manifold. A <u>variation of Hodge structure</u> consists of

(1) a $\mathbb{Q}$-local system $\underline{V}$ on S

(2) a decreasing filtration $F^p\mathcal{V}$ of $\mathcal{V} := \underline{V}\otimes_{\underline{\mathbb{C}}}\mathcal{O}_S$ satisfying Griffiths transversality

i.e. $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V}\otimes_{\mathcal{O}_S}\Omega^1_S$.

s.t. at each $s\in S$, this filtration gives a Hodge structure on $\underline{V}_s$.

injective

$\shortparallel$

Theorem Let (G,X) be a Shimura datum, & let $G\to GL(V)$ be a faithful $\mathbb{Q}$-rep'n of G.

(1) The part in (SV1) claiming that the Hodge types of $Ad\circ h:\mathbb{S}\to\mathfrak{g}_{\mathbb{R}}$ are of types $(-m,m)$ for some m

$\Rightarrow$ there's a unique complex structure on X s.t. the filtration on V_X defined by the $h:\mathbb{S}\to G_{\mathbb{R}}\to GL(V_{\mathbb{R}})$ varies holomorphically.

(2) Under (1), (SV1) (meaning the Hodge types $\in\{(-1,1),(0,0),(1,-1)\}$) is equivalent to that the filtration for the above Hodge structure satisfies Griffiths transversality

(3) (SV2) implies (when V has pure wt n) that there's a polarization

$\Psi: V\times V\to\mathbb{R}_X(-n)$ for the Hodge structure

Proof: (1) Will only prove existence.

The condition in (1) $\Rightarrow$ $h|_{\mathbb{G}_m}$ acts trivially on $\mathfrak{g}_{\mathbb{R}}$

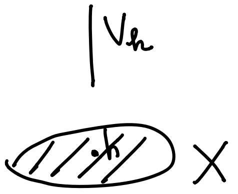
$\Rightarrow h(\mathbb{G}_m)\subseteq Z^0_{G_{\mathbb{R}}}$ ← connected component of center of $G_{\mathbb{R}}$.

May assume that V is irreducible $\Rightarrow$ $\mathbb{G}_m \xrightarrow{h} Z^0_{G_\mathbb{R}} \longrightarrow GL(V_\mathbb{R})$

acts by $x\mapsto x^{-n}$ for some n.

Each $h\in X$ defines a Hodge structure on $V=V_h$

$$\mathbb{S}\xrightarrow{h}G_\mathbb{R}\longrightarrow GL(V_\mathbb{R})$$



i.e. $V_{h,\mathbb{C}}=\bigoplus_{p+q=n}V_h^{pq}$ s.t. $h(z)$ acts on V_h^{pq} by $z^{-p}\bar{z}^{-q}$

Define $F^pV_{h,\mathbb{C}} := \bigoplus_{p'\geq p} V_h^{p'q}$ so that $V_h^{pq} = F^pV_{h,\mathbb{C}}\cap\overline{F^qV_{h,\mathbb{C}}}$

Easy to see, $\dim F^pV_{h,\mathbb{C}}$ is independent of h. (b/c doesn't change by $G(\mathbb{R})$-conj)

As each V_h is isomorphic to V abstractly, get a natural map

$\varphi: X \longrightarrow Gr(V_\mathbb{C}; \dim F^pV_{h,\mathbb{C}})$ = Grassmannian of filtrations on $V_\mathbb{C}$ with

$h \longmapsto F^pV_{h,\mathbb{C}}\subseteq V$ given dimension data.

Now, we verify that the image of X is a sub-analytic variety of the Grassmannian

E.g. $h: \mathbb{S} \to GL_2(\mathbb{R})$
$z = x+iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ $\rightsquigarrow \mathfrak{h}^{\pm} \subseteq G/B(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$

* The tangent space of the Grassmannian is $\mathrm{End}(V_{\mathbb{C}}) / \underbrace{F^0\mathrm{End}(V_{\mathbb{C}})}$

↑ those endom. preserving the filtration on $V_{\mathbb{C}}$

(Pf: e.g. $0 = F^2V_{\mathbb{C}} \subseteq F^1V_{\mathbb{C}} \subseteq F^0V_{\mathbb{C}} = V_{\mathbb{C}}$
$\qquad\qquad \overset{\shortparallel}{\mathbb{C}^m} \qquad \overset{\shortparallel}{\mathbb{C}^n} \qquad m<n$

deforming this filtration to $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ amounts to choosing
$\tilde{F}^1 \subseteq V_{\mathbb{C}} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$
↳ with basis $e_1 + (*e_{m+1} + \cdots + *e_n)\varepsilon$
$e_2 + (*e_{m+1} + \cdots + *e_n)\varepsilon$
$\ldots$

all the choices form canonically
$\mathrm{Hom}(F^1, V_{\mathbb{C}}/F^1V_{\mathbb{C}}) = \mathrm{End}(V_{\mathbb{C}})/F^0\mathrm{End}(V_{\mathbb{C}})$

The tangent space map of φ factors as

note: for a Hodge structure W, W^{00} is def'd over $\mathbb{R}$.

$$\begin{tikzcd}
T_hX = \mathfrak{g}/\mathfrak{g}^{00} \arrow[r] \arrow[d, "\cong"'] \arrow[dr, "d\varphi"] & \mathrm{End}(V)/\mathrm{End}(V)^{00} \arrow[d, phantom, "\cong"] \\
\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \arrow[r, "\text{injective when } G \to GL(V) \text{ is faithful.}"'] & \mathrm{End}(V)/F^0\mathrm{End}(V)
\end{tikzcd}$$

This defines a natural complex structure on X so that $X \to Gr$ is holomorphic.

Note: When choosing $V = (\mathfrak{g}, \mathrm{Ad})$, $d\varphi$ is an isomorphism
$X \hookrightarrow Gr$ is an open embedding.

(2) The Griffiths transversality translate to that the image of $d\varphi$ lies in $F^{-1}\mathrm{End}(V)/F^0\mathrm{End}(V)$
$\iff$ Hodge types on $(\mathfrak{g}, \mathrm{Ad}\circ h)$ can only be $(-1,1), (1,-1), (0,0)$.

(3) This is some results from Lie theory. Omit here.

§2. Classification of Shimura data

Given a homomorphism $h: \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$

$$\rightsquigarrow h_{\mathbb{C}} : (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)\times_{\mathbb{R}}\mathbb{C} \longrightarrow G_{\mathbb{C}}$$

$$\begin{matrix} \wr\wr \\ \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \end{matrix}$$

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \qquad z\otimes r$$

$$\wr\wr \qquad\qquad \downarrow$$

$$\mathbb{C}^\times \times \mathbb{C}^\times \qquad (zr, \bar{z}r)$$

Define

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}}$$

$$z \longmapsto (z, 1)$$

($G(\mathbb{C})$-conj. class) Hodge cocharacter attached to the Shimura datum (G, X)

Fact: There's a 1-1 correspondence between

$$\left\{\begin{matrix} G^{ad}(\mathbb{R})\text{-conjugacy classes of} \\ h : \mathbb{S} \to G_{\mathbb{R}} \text{ satisfying (SV1),(SV2)} \end{matrix}\right\} \hookrightarrow \left\{\begin{matrix} G(\mathbb{C})\text{-conjugacy classes of} \\ \mu : \mathbb{G}_m \to G_{\mathbb{C}} \text{ that are minuscule} \end{matrix}\right\}$$

small subtle point here: $G^{ad}(\mathbb{R})$-conjugacy class may contain more than one $G(\mathbb{R})$ conjugacy class

through $\quad G(\mathbb{R}) \to G^{ad}(\mathbb{R}) \to H^1(\mathbb{R}, Z_G)$

* minuscule: $\mu : \mathbb{G}_m \to G \rightsquigarrow \mu \in X_*(T)^{dom.}$ (by choosing a max'l torus)

$\rightsquigarrow$ Langlands dual group $\hat{G}$ s.t. $(X^*(\hat{T}), \hat{\Delta}, X_*(\hat{T}), \hat{\Delta}^\vee) = (X_*(T), \Delta, X^*(T), \Delta^\vee)$

Then $\mu \rightsquigarrow \mu \in X^*(\hat{T})^{dom} \rightsquigarrow$ highest weight rep'n $\hat{G} \to GL(V_\mu)$

Say μ is minuscule if all wts in V_μ have multiplicity 1.

- We say that (G, X) is of Hodge type if

  $\exists$ embedding $G \hookrightarrow GSp_{2g}$ s.t. $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \longrightarrow GSp_{2g,\mathbb{R}}$

  is conjugate to $z = x+iy \longmapsto \begin{pmatrix} xI_g & -yI_g \\ yI_g & xI_g \end{pmatrix}$

  i.e. Sh_G can be viewed as a moduli space of abelian varieties (with Hodge tensors

- We say that (G', X') is of abelian type if $\exists$ (G, X) of Hodge type s.t.

  * $\exists$ $G^{der} \to G'^{der}$ isog. inducing $G^{ad}(\mathbb{R})\cdot X \xrightarrow{\sim} G'^{ad}(\mathbb{R})\cdot X'$

  Basically, it means Hodge type up to center, except for one particular case.

Classification for simple G with μ. Type = Type of G

Type A: $\hat{G} = GL_{n/\mathbb{C}}$, $\mu = (\overbrace{1, \cdots 1}^{a}, 0, \cdots 0)$, $\hat{G} \xrightarrow{\wedge^a} GL_{\binom{n}{a}}(\mathbb{C})$ Hodge type

$G = U(a, n-a) \qquad h : \mathbb{S} \to G_{\mathbb{R}} \qquad a$

$z \mapsto \mathrm{diag}\{z/\bar{z}, \cdots, z/\bar{z}, 1, \cdots, 1\}$

Type B $\hat{G} = GSp_{2n}$, $\mu \leftrightarrow$ vector rep'n spin rep'n. Hodge type

$G = GSpin(2, 2n-1) \hookrightarrow GSp_{2^{2n}}$

Type C,D $\hat{G} = GSpin_n/\mathbb{C}$, $\mu \leftrightarrow$ spin rep'n of $\hat{G}$ Both Hodge type

$$G = \begin{cases} GSp(n-1) & n \text{ odd} \leftarrow \text{Siegel case.} \\ GSpin(2, n-2) & n \text{ even} \leftarrow \text{more complicated} \end{cases}$$

$\hookrightarrow GSp_{2^{n-2}}$

Type D^H $\hat{G} = SO_{2n}$ vector rep'n.

$G = GSO_{2n} \hookrightarrow GSp_{4n}$ Hodge type

But if G is $\mathrm{Res}_{F/\mathbb{Q}} G_0$ for F totally real. with G_0 simple of type D.

then for each embedding $G \times_{F,\tau} \mathbb{R}$ is of either type D or D^H.

If we want G to be of abelian type, we need the type above has to be uniform everywhere

(plus another condition in case D^H: G_0^{der} has to a quotient of SO_{2n}.)

Type E_6, E_7, $\exists$ minuscule rep'ns but not of abelian type.

§3 Shimura reciprocity law and canonical model

* Reflex field : $(G, X) \rightsquigarrow$ a $G(\mathbb{C})$-conjugacy class of cocharacters $\mu: \mathbb{G}_m \to G_{\mathbb{C}}$

$\{\mu: \mathbb{G}_m \to G_{\mathbb{C}}\}$ is a natural variety, can be defined over a number field $E \subseteq \mathbb{C}$

E is called the reflex field.

Explicitly, $\underset{\substack{\cup\!\shortmid \\ \mu}}{X_*(T_{\mathbb{C}})^{dom}} \circlearrowleft \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$ where $\mathbb{Q}^{alg}$ = alg. closure of $\mathbb{Q}$ inside $\mathbb{C}$

Then $E = E(G, X)$ = subfield of $\mathbb{Q}^{alg}$ fixed by $\mathrm{Stab}_{\mu}(\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}))$

Note: The reflex field is always a subfield of $\mathbb{C}$! i.e. a number field with a specific cplx embed'g.

E.g. $\begin{array}{c} E \\ | \\ \mathbb{Q} \end{array}$ V n-dim'l Herm space $/E$. of signature $(a, n-a)$

Then $GU(V) \times_{\mathbb{Q}} \mathbb{C} \simeq GL_n \times \mathbb{G}_m$, $h: z \mapsto \mathrm{diag}(z, \cdots, z, \bar{z}, \cdots, \bar{z})$

$$X_*(T_n \times \mathbb{G}_m) \cong \mathbb{Z}^n \times \mathbb{Z} \ni \mu = (\overbrace{1 \cdots 1}^{a}, \overbrace{0, \cdots, 0}^{n-a}; 1)$$

$$\circlearrowleft \mathrm{Gal}(E/\mathbb{Q}) \ni \sigma: (a_1, \cdots, a_n; b) \mapsto (b - a_n, \cdots, b - a_1; b)$$

Then: if $a = n-a$, μ is invariant under σ $\Rightarrow$ reflex field is $\mathbb{Q}$

if $a \neq n-a$, the reflex field is $E \subseteq \mathbb{C}$

E.g. $F/\mathbb{Q}$ totally real. HMV: $G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$,

$$h : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G_{\mathbb{R}} = \prod_{\mathrm{Hom}(F,\mathbb{R})} GL_2(\mathbb{R})$$

$$z = x+iy \longmapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$$\mu : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}} = \prod_{\mathrm{Hom}(F,\mathbb{R})} GL_{2,\mathbb{C}}$$

$$z \longmapsto \begin{pmatrix} z & \\ & 1 \end{pmatrix}$$

$X_*(\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m^2) \simeq \bigoplus_{\mathrm{Hom}(F,\mathbb{C})} \mathbb{Z}^2 \circlearrowleft \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$ permuting the embeddings

$\mu \longmapsto ((1,0))$ on every component. $\Rightarrow$ fixed by all $\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

$\Rightarrow$ reflex field is $\mathbb{Q}$.

Theorem. The tower of Shimura variety $Sh(G,X) = (Sh_K(G,X))_{K \subseteq G(\mathbb{A}_f)}$

$$\left(Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \right)$$

admits a canonical model over the reflex field $E = E(G,X)$

$\uparrow$ now, explain this.

* If $G = T$ is a torus (i.e. $T_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}}^n$), $h : \mathbb{S} \to T_{\mathbb{R}}$ is invariant under conjugation

$\Rightarrow X = \{h\}$ is a singleton

& $\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ is def'd over the reflex field $E \subseteq \mathbb{C}$.

$\rightsquigarrow$ For $K \subseteq T(\mathbb{A}_f)$ open compact, $Sh_K(T)(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$ is a finite set.

To define a model of $Sh_K(T)$ over E, it's enough to specify $\mathrm{Gal}(\mathbb{Q}^{alg}/E)$-action.

Shimura reciprocity map: E = reflex field

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}^{alg}/E) \arrow[r, two heads] \arrow[d, bend right] & \mathrm{Gal}(E^{ab}/E) & E^\times \backslash \mathbb{A}_E^\times / E_{\mathbb{R}}^{\times,0} \arrow[l, "\mathrm{Art}"', "\cong"] \arrow[r] & \mathbb{G}_{m,E}(\mathbb{Q}) \backslash \mathbb{G}_{m,E}(\mathbb{A}_f) \arrow[d, "\mu_E"] \\
 & & & {}
\end{tikzcd}$$

$\uparrow$ sending uniformizer to geom. Frob.

$$\begin{tikzcd} \arrow[rr, bend right, "Rec_\mu"'] & T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f) \arrow[r, "Nm_{E/\mathbb{Q}}"] & T(\mathbb{Q})\backslash T(\mathbb{A}_f) \end{tikzcd}$$

The canonical model of $Sh_K(T)$ over Spec E is the E-scheme structure s.t. the induced $\tau \in Gal(\mathbb{Q}^{alg}/E)$ on $Sh_K(T)(\mathbb{C})$ is given by right translate by $Res_\mu(\tau)$.

- For general (G,X), a canonical model of the Shimura variety is an E-scheme $Sh_K(G,X)$ s.t. for every morphism $(T,\{h\}) \to (G,X)$ of Shimura data, the natural morphism $T(\mathbb{Q})\backslash \{h\} \times T(\mathbb{A}_f) / K \cap T(\mathbb{A}_f) \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f) / K$

  is induced by a morphism $Sh_{K\cap T(\mathbb{A}_f)}(T,\{h\}) \longrightarrow Sh_K(G,X) \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, E(T,\{h\})$

(Basically, there are "enough" such $(T,\{h\}) \to (G,X)$ to rigidify the scheme structure of $Sh_K(G,X)$.)