Dual BGG complexes

§1. Quick introduction to category $\mathcal{O}$.

Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{Mat}_2(\mathbb{C})^{tr=0}$ be a reductive Lie algebra / $\mathbb{C}$

$\cup|$ ↰ will use this color to indicate this example

$\mathfrak{h} = \begin{pmatrix} * & \\ & * \end{pmatrix}$ be the Cartan subalg

$$\mathfrak{h} \text{ acts on } \mathfrak{g} \Rightarrow \mathfrak{g} = \underset{\mathfrak{g}_0}{\underset{\shortparallel}{\mathfrak{h}}} \oplus \bigoplus_{\substack{\alpha\in\Phi \\ \cap| \\ \mathrm{Hom}(\mathfrak{h},\mathbb{C})}} \mathfrak{g}_\alpha = \underset{\mathfrak{g}_0}{\underset{\shortparallel}{\left\{\begin{pmatrix} a & \\ & -a\end{pmatrix}\right\}}} \oplus \underset{\mathfrak{g}_\alpha}{\underset{\shortparallel}{\begin{pmatrix}0&*\\0&0\end{pmatrix}}} \oplus \underset{\mathfrak{g}_{-\alpha}}{\underset{\shortparallel}{\begin{pmatrix}0&0\\ *&0\end{pmatrix}}}$$

s.t. $\forall\ x\in\mathfrak{g}_\alpha,\ h\in\mathfrak{h},\ [h,x] = \alpha(h)\cdot x.$

Φ is called the set of roots of Lie algebra $\mathfrak{g}=\mathfrak{gl}_2$

$$\Phi = \underset{\uparrow\text{positive roots}}{\Phi^+} \sqcup \underset{\uparrow\text{negative roots}}{\Phi^-} = \{\alpha\} \sqcup \{-\alpha\}$$

- We will only consider (possibly infinite dim'l) algebraic rep'ns V of $\mathfrak{g}$, on which $\mathfrak{h}$ acts semisimply.

i.e. $V = \bigoplus_{\chi\in\mathrm{Hom}(\mathfrak{h},\mathbb{C})} V_\chi$, $V_\chi = \{v\in V,\ hv = \chi(h)v\}$

↑ called the weights of V.

$\forall\ e_\alpha \in \mathfrak{g}_\alpha$, then $e_\alpha : V_\chi \longrightarrow V_{\chi+\alpha}$

$$\text{b/c}\quad h(e_\alpha v) = e_\alpha h v + [h, e_\alpha] v$$
$$= \chi(h)\cdot e_\alpha v + \alpha(h) e_\alpha v \quad ✓$$

E.g. $\mathfrak{sl}_2$. typical V looks like

$$\begin{tikzcd}
\cdots & V_{\chi-\alpha} \arrow[r, bend left, "e_\alpha"] & V_\chi \arrow[l, bend left, "f_\alpha"] \arrow[r, bend left, "e_\alpha"] & V_{\chi+\alpha} \arrow[l, bend left, "f_\alpha"] \arrow[r, bend left, "e_\alpha"] & V_{\chi+2\alpha} \arrow[l, bend left, "f_\alpha := e_{-\alpha}"] & \cdots
\end{tikzcd}$$

- Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

i.e. $\mathcal{U}(\mathfrak{g}) = \mathbb{C}\langle x\,;\ x\in\mathfrak{g}\rangle / (xy - yx - [x,y])$

↑ non-comm. alg generated by $\mathfrak{g}$.

Then algebraic rep'n of $\mathfrak{g}$ = algebraic rep'n of $\mathcal{U}(\mathfrak{g})$.

- Let $\mathfrak{q} \subseteq \mathfrak{g}$ be a parabolic subalgebra of $\mathfrak{g}$. ($\mathfrak{q}$ is a Lie subalg containing) e.g. $\mathfrak{q} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathfrak{sl}_2$. $\quad$ $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$)

$\rightsquigarrow \mathcal{U}(\mathfrak{q}) \subseteq \mathcal{U}(\mathfrak{g})$ a subalgebra.

Def'n: The category $\mathcal{O}$ for $(\mathfrak{q}, \mathfrak{g})$ is the category of $\mathcal{U}(\mathfrak{g})$-rep'ns V s.t.

(1) V is finitely generated

(2) $\mathfrak{h}$ acts semisimply on V

(3) $\forall v \in V$, the $\mathbb{C}$-vector space $\mathcal{U}(\mathfrak{q})\cdot v$ is finite dim'l

(as we go up $V_\chi \xrightarrow{e_\alpha} V_{\chi+\alpha} \xrightarrow{e_\alpha} V_{\chi+2\alpha} \cdots$ it becomes zero at some point)

Morphisms are $\mathcal{U}(\mathfrak{g})$-morphisms

E.g. • All finite dimensional rep'ns of $\mathfrak{g}$. $\quad$ e.g. $\overset{\text{standard rep'ns}}{\mathrm{Sym}^r \mathbb{C}^2} = \underbrace{V_{-\frac{r}{2}\alpha} \oplus V_{(-\frac{r}{2}+1)\alpha} \oplus \cdots \oplus V_{\frac{r}{2}\alpha}}_{\text{all 1-dim'l}}$

$$\begin{tikzcd} V_{-\frac{r}{2}\alpha} \arrow[r, bend right, "e_\alpha"'] & V_{(-\frac{r}{2}+1)\alpha} \arrow[r, bend right, "e_\alpha"'] & \cdots \arrow[r, bend right, "e_\alpha"'] & V_{\frac{r}{2}\alpha} \end{tikzcd}$$

• For a finite dim'l rep'n W of $\mathfrak{q}$, $\rightsquigarrow V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W$

Explicitly, write $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{n}^-$ $\leftarrow$ opposite nilpotent Lie algebra

Then $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \cdot \mathcal{U}(\mathfrak{q}) \Rightarrow V = \mathcal{U}(\mathfrak{n}^-) \cdot W$

$\uparrow$ as $\mathbb{C}$-vector space, not as algebra, respect $\mathfrak{h}$-weight

e.g. $\mathfrak{q} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, take $\lambda : \begin{pmatrix} \mathbb{C}^\times & * \\ 0 & \mathbb{C}^\times \end{pmatrix} \longrightarrow \mathbb{C}^\times$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \lambda_1(a)\, \lambda_2(d)$$

$\mathfrak{n}^- = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, $\mathcal{U}(\mathfrak{n}^-) = \mathbb{C}[X]$

Then $\quad \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \lambda = V_\lambda \oplus V_{\lambda-\alpha} \oplus V_{\lambda-2\alpha} \oplus \cdots$ $\quad$ infinite dim'l.

Adjunction: $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W, V) = \mathrm{Hom}_{\mathcal{U}(\mathfrak{q})}(W, V|_{\mathcal{U}(\mathfrak{q})})$

E.g. There's a natural morphism $n \geqslant 0$

$$\underbrace{\mathcal{U}(\mathfrak{sl}_2) \otimes_{\mathcal{U}(\mathfrak{b})} \left(\tfrac{n}{2}\alpha\right)}_{} \longrightarrow \underbrace{\mathrm{Sym}^n \mathbb{C}^2}_{}$$

b/c $\frac{n}{2}\alpha \xrightarrow{\mathcal{U}(\mathfrak{b})} \mathrm{Sym}^{n\alpha} \mathbb{C}^2$ as highest weight.

$\uparrow$ ∞-dim'l $\quad$ ¦ $\quad$ fin. dim'l, irred.

weights are $\quad \cdots, (\frac{n}{2}-1)\alpha, \frac{n}{2}\alpha$ $\quad$ ¦ $\quad$ $-\frac{n}{2}\alpha, (-\frac{n}{2}+1)\alpha, \cdots, \frac{n}{2}\alpha$

So the kernel has weights $\quad \cdots (-\frac{n}{2}-2)\alpha, (-\frac{n}{2}-1)\alpha$.

Fact: there's an exact sequence

$$0 \to \mathcal{U}(sl_2) \otimes_{\mathcal{U}(b_2)} \left(\left(-\tfrac{n+2}{2}\right)\alpha\right) \to \mathcal{U}(sl_2) \otimes_{\mathcal{U}(b_2)} \left(\tfrac{n}{2}\alpha\right) \to \mathrm{Sym}^{n\alpha}\mathbb{C}^2 \to 0$$

This is the $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\text{BGG resolution}}$ for sl_2.

Another viewpoint: Consider the center $Z(g) := Z(U(g))$ action

$$g = sl_2 = \left\langle E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \qquad \swarrow [E,F] = H$$

$$C = EF + FE + \tfrac{1}{2}H^2 \in Z(g) \quad \text{Casimir operator} \qquad C = 2FE + H + \tfrac{1}{2}H^2$$

Fact: C acts by mult. by $2\lambda^2 + 2\lambda$ on $\mathcal{U}(sl_2) \otimes_{\mathcal{U}(b_2)} (\lambda\alpha)$

Suffices to consider the C-action on highest weight vector $e_{\lambda\alpha}$

$$\left(2FE + \left(H + \tfrac{1}{2}H^2\right)\right) e_{\lambda\alpha} = \underset{\substack{\uparrow \\ \text{as } e_{\lambda\alpha} \text{ is highest wt}}}{0} + \left(2\lambda + \tfrac{1}{2}(2\lambda)^2\right) e_{\lambda\alpha} = \left(2\lambda^2 + 2\lambda\right) e_{\lambda\alpha}$$

$\Rightarrow$ $\mathcal{U}(g) \otimes_{\mathcal{U}(b)} \left(\left(-\frac{n+2}{2}\right)\alpha\right)$ & $\mathcal{U}(g) \otimes_{\mathcal{U}(b)} \left(\frac{n}{2}\alpha\right)$ are the only two Verma modules with C-eigenvalue $\frac{1}{2}n^2 + n$.

(i.e. no maps/extensions among Verma modules w/ different C-eigenvalues)

Borel version of BGG resolution for gl_n (optional)

$$g = gl_n \supseteq b = b_n = \begin{pmatrix} * & * \\ 0 & \ddots \ * \end{pmatrix} \supseteq h = h_n = \begin{pmatrix} * & 0 \\ 0 & \ddots \ * \end{pmatrix}$$

$\mathrm{Hom}(h, \mathbb{C}) \cong \{\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n,\ \lambda_i \in \mathbb{C}\}$ α_i = evaluation of (i,i)th entry.

$$g = g_0 \oplus \bigoplus_{\alpha \in \Phi} g_\alpha = \underset{\substack{\parallel \\ g_0}}{h} \oplus \bigoplus_{i \neq j} \underset{\substack{\parallel \\ g_{\alpha_i - \alpha_j}}}{\left(\quad * \leftarrow i,j \text{ entry} \right)}$$

So $\Phi = \{\alpha_i - \alpha_j,\ \alpha \neq j\} \supseteq \Phi^+ = \{\alpha_i - \alpha_j,\ i > j\}$.

* Simple roots $\Delta = \{\underset{\substack{\updownarrow \\ s_1}}{\alpha_1 - \alpha_2}, \underset{\substack{\updownarrow \\ s_2}}{\alpha_2 - \alpha_3}, \ldots, \underset{\substack{\updownarrow \\ s_{n-1}}}{\alpha_{n-1} - \alpha_n}\}$, all positive roots are $\underline{\text{positive}}$ linear comb. of Δ

reflection of $\mathrm{Hom}(h, \mathbb{C})$

$$s_i(\lambda_1,\cdots,\lambda_i,\lambda_{i+1},\cdots,\lambda_n)=(\lambda_1,\cdots,\lambda_{i+1},\lambda_i,\cdots,\lambda_n)$$

$\rightsquigarrow$ s_i's generate a subgroup : Weyl group $W=S_n \hookrightarrow \mathrm{Hom}(\mathfrak{h},\mathbb{C})$ keeping $\Phi\subseteq\mathrm{Hom}(\mathfrak{h},\mathbb{C})$ stable.

But, we need a slight twist of this action (called $*$-action)

$$s_i*(\lambda_1,\cdots,\lambda_i,\lambda_{i+1},\cdots,\lambda_n)=(\lambda_1,\cdots,\lambda_{i+1}-1,\lambda_i+1,\cdots,\lambda_n)$$

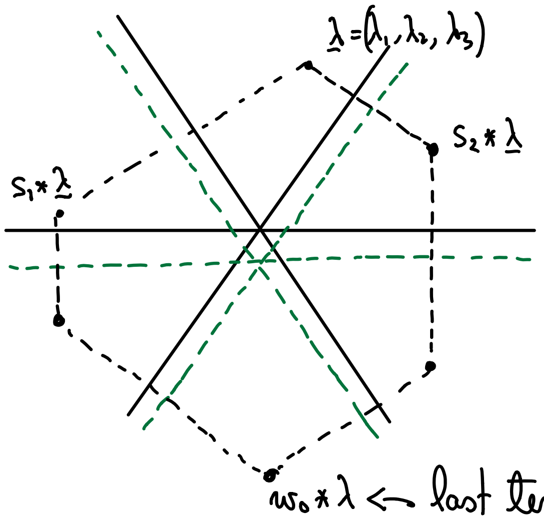
(same as $s_i\left(\underline{\lambda}+\left(\frac{n-1}{2},\frac{n-3}{2},\cdots,-\frac{n-1}{2}\right)\right)-\left(\frac{n-1}{2},\frac{n-3}{2},\cdots,-\frac{n-1}{2}\right)$) $\leftarrow$ $\rho=\frac{1}{2}\sum(\text{all positive roots})$

$s*(n,0)=(-1,n-1)$ <u>note</u> : $\frac{1-(n-1)}{2}=-\frac{n-2}{2}$

For a dominate integral weight $\underline{\lambda}=(\lambda_1,\cdots,\lambda_n)\in\mathrm{Hom}(\mathfrak{h},\mathbb{C})$, $\lambda_i\in\mathbb{Z}$ & $\lambda_1>\cdots>\lambda_n$ there's a unique finite dimensional rep'n $V_{\underline{\lambda}}$ of $\mathfrak{gl}_n$.

The BGG resol'n for $\mathfrak{gl}_3$ is

$$\cdots\longrightarrow\bigoplus_{\substack{w\in S_3\\ \text{length } 2}}\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_{w*\underline{\lambda}}\longrightarrow\bigoplus_{i=1}^{2}\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_{s_i*\underline{\lambda}}\longrightarrow\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_{\underline{\lambda}}\longrightarrow V_{\underline{\lambda}}\longrightarrow 0$$



$\leftarrow$ last term in the BGG resolution.

Will use a parabolic version.

§2. <u>Automorphic vector bundles over Shimura varieties</u>

- <u>Fact</u>: $G(\mathbb{Q})\backslash\left(\left(G(\mathbb{R})/K_\infty\right)\times\left(G(\mathbb{A}_f)/K_f\right)\right)\simeq\coprod_i\Gamma_i\backslash G(\mathbb{R})/K_\infty$

for Γ_i some discrete arithmetic subgps of $G(\mathbb{Q})$.

<u>Today</u>: Will only focus on $\Gamma\backslash G/K$ with

G semisimple real group

G/K Hermitian symmetric domain.

Γ some discrete subgp of G

- Built-in for the theory of Hermitian symmetric domain

$\mathfrak g = \mathrm{Lie}\, G$, $\theta : G \to G$ is a Cartan involution

$K = G^{\theta}$ = max'l compact subgroup

$$\mathfrak g = \mathfrak g^{\theta=1} \oplus \mathfrak g^{\theta=-1} = \mathfrak k \oplus \mathfrak p$$

Fact: $G \overset{\text{homeo.}}{\simeq} K \times \exp(\mathfrak p)$

- $h : \mathbb C^{\times} \longrightarrow G$ Deligne's homo. with $\theta = \mathrm{Ad}_{h(i)}$

By (SV1),
$$\mathfrak g_{\mathbb C} \cong \underset{\mathfrak p^-}{\underset{\|}{\underbrace{\mathfrak g^{1,-1}}}} \oplus \underset{\mathfrak k_{\mathbb C}}{\underset{\wr\|}{\mathfrak g^{0,0}}} \oplus \underset{\mathfrak p^+}{\underset{\|}{\mathfrak g^{-1,1}}}$$

Sign convention: always assume $\mu : \mathbb G_m \to G_{\mathbb C}$ to be dominant

$\Rightarrow \mu(\mathbb G_m)$ acts on $\mathfrak g^{1,-1}$ by z^{-1}

$\mathfrak g^{-1,1}$ by z

b/c Deligne says $h(z)$ acts on $\mathfrak g^{p,q}$ by $z^{-p}\bar z^{-q}$.

$$\mathfrak p_{\mathbb C} \cong \mathfrak p^- \oplus \mathfrak p^+$$

Then $\mathfrak p^{\pm}$ are stable under K-action

& $[\mathfrak p^+, \mathfrak p^+] = [\mathfrak p^-, \mathfrak p^-] = 0$, $[\mathfrak p^+, \mathfrak p^-] \subseteq \mathfrak k_{\mathbb C}$

* Fix a Cartan subgroup $H \subseteq K$

$$\mathfrak g_{\mathbb C} = \mathfrak h_{\mathbb C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak g_{\alpha} \quad \text{with } \alpha \in i\, \overset{\mathrm{Hom}(\mathfrak h, \mathbb R)}{\overset{\|}{\mathfrak h^*}}$$

Fact: $\Phi = \Phi_c \sqcup \Phi_{nc}$ ← noncompact.

$$\mathfrak k_{\mathbb C} = \mathfrak h_{\mathbb C} \oplus \bigoplus_{\alpha \in \Phi_c} \mathfrak g_{\alpha}, \quad \mathfrak p_{\mathbb C} = \bigoplus_{\alpha \in \Phi_{nc}} \mathfrak g_{\alpha}$$

$G = SL_2$, $\theta : X \longmapsto \mathrm{Ad}_{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}(X)$

$K = G^{\theta} = SO_2$

char poly $= x^2+1$
eigenvalues $\pm i$

$$\mathfrak{sl}_2 = \mathfrak{sl}_2^{\theta=1} \oplus \mathfrak{sl}_2^{\theta=-1}$$

$\mathfrak k = \mathbb R \cdot \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$ $\quad$ $\mathfrak p = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\}$

$h : \mathbb C^{\times} \longrightarrow SL_2$ $\quad$ $h(i) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$z = x+iy \longmapsto \frac{1}{\sqrt{x^2+y^2}} \boxed{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}}$$

← This seems to be the correct normalization

$\sqrt{x^2+y^2} \cdot h(x+iy)$ has eigen vectors $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ w/ eigenval $x+iy$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$ w/ eigenval $x-iy$

$$\mathfrak{sl}_{2,\mathbb C} = \underset{\|}{\overset{\mathfrak g^{0,0}}{}}\mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} ic & \\ & -ic \end{pmatrix} \oplus \underbrace{\mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}}_{\text{eigenval. } \frac{x+iy}{x-iy}} \oplus \underbrace{\mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}}_{\text{eigenval } \frac{x-iy}{x+iy}}$$

$\mathfrak g^{0,0}$ $\quad$ $\mathfrak p^+ = \mathfrak g^{-1,1}$ $\quad$ $\mathfrak p^- = \mathfrak g^{1,-1}$

$\mathfrak h_{\mathbb C} = \mathfrak k_{\mathbb C}$, $\mathfrak g_{\alpha} = \mathfrak p^+$, $\mathfrak g_{-\alpha} = \mathfrak p^-$.

$$\alpha : \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = \mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} ic & \\ & -ic \end{pmatrix} \longmapsto 2ic$$

$\Phi = \Phi_{nc} = \{\pm\alpha\}$, $\Phi^+ = \{\alpha\}$

Write $\Phi = \Phi^+ \sqcup \Phi^-$ so that

$$\mathfrak{p}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}_{nc}} \mathfrak{g}_\alpha \quad \text{where } \Phi^{\pm}_{nc} = \Phi_{nc} \cap \Phi^{\pm}$$

⇝ Lie algebra $\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- \subseteq \mathfrak{g}_{\mathbb{C}}$

⇝ parabolic subgroup Q

$$\begin{tikzcd} D = G/K \arrow[r] \arrow[rrd, bend right, "\text{is an open immersion}"'] & G_{\mathbb{C}}/K_{\mathbb{C}} \arrow[r, two heads] & G_{\mathbb{C}}/Q =: \check{D} \arrow[d, phantom, "\cup" rotate=90] \\ & & \exp(\mathfrak{p}^+) \cong \exp(\mathfrak{p}^+)\cdot Q/Q \end{tikzcd}$$

$W = $ Weyl group $\supseteq W_c = $ Weyl group of Φ_c

$\cup$

$W_n = \{$representative of W/W_c with minimal length$\}$

$= \{w \in W$, sending G-dom. weights to K-dom.$\}$

Given a rep'n of Q: $Q \to GL(V)$ V fin.dim$/\mathbb{C}$

⇝ a vector bundle

$$\begin{tikzcd} G_{\mathbb{C}} \arrow[d] & G_{\mathbb{C}} \times V \arrow[l] \arrow[d] & \\ \check{D} = G_{\mathbb{C}}/Q & (G_{\mathbb{C}} \times V)/Q \arrow[l] & =: \underline{V} \end{tikzcd}$$

⇝ $\underline{V}|_D$ & then take quotient by Γ

→ $\underline{V}$ on $Sh_G = \Gamma \backslash D / K_\infty$

- $T_{\check{D}} \cong \underline{(\mathfrak{g}/\mathfrak{p}^-)}$ (for adjoint Q-action)

$\Omega^1_{\check{D}} \cong \Omega^1_{G_{\mathbb{C}}/Q} = \underline{\mathfrak{p}^-}$ (for the adjoint Q-action)

$\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- = \mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

⇝ $Q = \mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} * & \\ * & * \end{pmatrix}$

$$\begin{tikzcd} D = SL_2(\mathbb{R})/SO_2 \arrow[r] \arrow[d, equal] & SL_2(\mathbb{C})/K_{\mathbb{C}} \arrow[r, "\cdot\begin{pmatrix} -i & i \\ i & 1 \end{pmatrix}"] & SL_2(\mathbb{C})/\begin{pmatrix} * & \\ * & * \end{pmatrix} \arrow[d, equal, "{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [b:d]}"'] \\ \mathfrak{h} \arrow[rr] & & \check{D} = \mathbb{P}^1 \end{tikzcd}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} * & ai+b \\ * & ci+d \end{pmatrix} \mapsto \dfrac{ai+b}{ci+d}$

$\mathbb{Z}/2\mathbb{Z}$

$\|$

$W = W_n = \{1, s_\alpha\}$

- G-dominant: $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longmapsto a^m \cdot d^n$ $m \geq n$

K-dominant any $(m,n) \in \mathbb{Z}^2$

$Q = \mathrm{Ad}_{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \cong \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \xrightarrow{(m,n)} \mathbb{C}^\times$

$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto a^m d^n$

⇝ a line bundle on $\check{D}$ & on Sh_G

It is $\omega^{n-m} \otimes \left(\wedge^2 \mathcal{H}^1_{dR}(E/Sh_G)\right)^{\otimes -n}$

(There's a "duality issue, V ⟷ homology theory)

$\mathfrak{p}^-$ corresponds to $m=-1$, $n=1$

$$\longleftrightarrow \Omega^1_{Sh_G} \cong \omega^{\otimes 2} \otimes \underbrace{\left(\wedge^2 \mathcal{H}^1_{dR}(E/Sh)\right)}_{\substack{\uparrow \\ \text{trivial bundle so people} \\ \text{usually ignore.}}}{}^{\otimes(-1)} \text{ on } Sh_G$$

<u>Big Theorem</u> Given W_1, W_2 two representations of Q

$\{$ G-equivariant differential maps $\underline{W}_1 \to \underline{W}_2$ on $\check{D}$ $\}$

$\mathrel{\rotatebox{90}{$\cong$}}$

Hom-Tensor adjunction

$$\mathrm{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{q})} W_2^*, U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{q})} W_1^*\right) \cong \mathrm{Hom}_{\mathfrak{q}}\left(W_2^*, U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{q})} W_1^*\right)$$

Example: Recall: $U(\mathfrak{sl}_2)\otimes_{\underline{\underline{U(\mathfrak{q})}}} x^{2+k} \longrightarrow U(\mathfrak{sl}_2)\otimes_{U(\mathfrak{q})} x^{-k}$ is the non-triv. map earlier

note that $\mathfrak{q}$ is the opposite Borel so, all weights are inverse of §1.

$\updownarrow$ corresponds to

$$\begin{tikzcd} \underline{x}^{k} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \underline{x}^{-2-k} \arrow[d, phantom, "\shortparallel"] \\ \omega^{-k} \arrow[r] & \omega^{k+2} \end{tikzcd}$$

θ-operator const. $\left(q\frac{d}{dq}\right)^{k+1}$.

Proof: First consider G-equivariant coherent sheaf maps:

$$\mathrm{Hom}_{\check{D}}(\underline{W}_1, \underline{W}_2) = \Gamma\left(G_{\mathbb{C}}/Q, \underbrace{\underline{W}_2\otimes_{\mathcal{O}_{\check{D}}}\underline{W}_1^*}_{\shortparallel\ \underline{W_2\otimes W_1^*}}\right)$$

But we want also left G-equiv $\rightsquigarrow$ need $(W_2\otimes W_1^*)^{\mathfrak{q}} = \mathrm{Hom}_{\mathfrak{q}}(W_2^*, W_1^*)$

Next consider G-equivariant first order differential operators

$$\mathrm{Hom}_{\check{D}}(\underline{W}_1\otimes\Omega^1_{\check{D}}, \underline{W}_2) = \Gamma\left(G_{\mathbb{C}}/Q, \underline{W_2\otimes W_1^*\otimes \mathfrak{p}^{-*}}\right)$$

So the G-equivariant ones are $(W_2\otimes W_1^*\otimes\mathfrak{p}^{-*})^{\mathfrak{q}} = \mathrm{Hom}_{\mathfrak{q}}(W_2^*, W_1^*\otimes\mathfrak{p}^+)$

Next: second order diff. operator

$$\mathrm{Hom}_{\check{D}}\left(\underline{W}_1\otimes(\Omega^1_{\check{D}})^{\otimes 2}, \underline{W}_2\right) \overset{G\text{-equiv.}}{\underset{\text{ones}}{\rightsquigarrow}} \mathrm{Hom}_{\mathfrak{q}}\left(W_2^*, W_1^*\otimes(\mathfrak{p}^+)^{\otimes 2}\right)$$

third order .... $\rightsquigarrow (\mathfrak{p}^+)^{\otimes 3}\ldots$

more subtle work to compare $\mathfrak{q}$-action.

Taking direct sum $\Rightarrow \mathrm{Hom}_{\mathfrak{q}}\left(W_2^*, W_1^*\otimes U(\mathfrak{p}^+)\right) = \mathrm{Hom}_{\mathfrak{q}}\left(W_2^*, U(\mathfrak{g}_{\mathbb{C}})\otimes_{U(\mathfrak{q})} W_1^*\right)$

§3 Dual BGG resolution

Theorem. Let λ denote a dominant weight of $G \rightarrow V(\lambda)$ highest weight rep'n of G

On $\Gamma\backslash D \rightsquigarrow \underline{V(\lambda)} := \Gamma\backslash\!\backslash\left(V(\lambda)\times(G_{\mathbb{C}}/Q)\right)$

locally constant $\mathbb{C}$-sheaf.

Then there's a resolution automorphic vector bundles on $\Gamma\backslash D$

$$0 \to \underline{V}(\lambda) \to \underline{K}^0_\lambda \to \underline{K}^1_\lambda \to \cdots$$

coming from G-equiv vector bundle on $\check{D} = G_{\mathbb{C}}/Q$

where $\underline{K}^i_\lambda = \bigoplus_{\substack{w\in W_n \\ \ell(w)=i}} \underline{W}\big(w(\lambda+\rho)-\rho\big)$ & the maps are differential operators.

Corollary: (When $\Gamma\backslash D$ is proper), $\exists$ a spectral sequence

$$E_1^{p,q} = \underbrace{H^q(\Gamma\backslash D, \underline{K}^p_\lambda)}_{\text{coherent cohomology}} \Rightarrow H^{p+q}_{Betti}\big(\Gamma\backslash D, \underline{V}(\lambda)\big)$$

Fact: This spectral sequence degenerate in E_1

Example:
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \end{tikzcd}$$
ignore the cusp issue $\mathcal{E} = R^1\pi_*\underline{\mathbb{C}}$ $k \geq 3$

$$0 \to \mathrm{Sym}^{k-2}\mathcal{E} \to \omega^{2-k} \to \omega^k \to 0$$

$$\boxed{\begin{matrix} H^1(X,\omega^{2-k}) \longrightarrow \cancel{H^1(X,\omega^k)} \\ \cancel{H^0(X,\omega^{2-k})} \longrightarrow H^0(X,\omega^k) \end{matrix}} \Rightarrow H^\bullet_{Betti}(X, \mathrm{Sym}^{k-2}\mathcal{E})$$

$$\rightsquigarrow 0 \to H^0(X,\omega^k) \to H^1_{Betti}(X,\mathrm{Sym}^{k-2}\mathcal{E}) \to \underset{\substack{\wr\wr \\ H^0(X,\omega^k)^\vee}}{\underline{H^1(X,\omega^{2-k})}} \to 0$$

Example: $F/\mathbb{Q}$ totally real

$$\begin{tikzcd} A \arrow[loop right, "\mathcal{O}_F"] \arrow[d, "\pi"] \\ X = HMV \end{tikzcd}$$

$R^1\pi_*\underline{\mathbb{C}} = \bigoplus_{\tau\in \mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{C})} \mathcal{E}_\tau$ ← rank 2 $\mathbb{C}$-sheaves on X

$$\begin{matrix} \omega = \bigoplus_{\tau\in\mathrm{Hom}(F,\mathbb{C})} \omega_\tau \\ \cap| \\ H^1_{dR} = \bigoplus_{\tau\in\mathrm{Hom}(F,\mathbb{C})} H^1_{dR,\tau} \end{matrix}$$

$(\underline{k},w) = \big((k_\tau)_{\tau\in\mathrm{Hom}(F,\mathbb{C})}, w\big) \in \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})} \times \mathbb{Z}$, $k_\tau \equiv w \pmod 2$

$$\omega^{(\underline{k},w)} = \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{C})} \omega_\tau^{k_\tau} \otimes \big(\wedge^2 H^1_{dR,\tau}\big)^{\otimes \frac{w-k_\tau}{2}}$$

$$\mathcal{E}^{(\underline{k},w)} = \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{C})} \mathrm{Sym}^{k_\tau-2}\mathcal{E}_\tau \otimes \big(\wedge^2\mathcal{E}_\tau\big)^{\frac{w-k_\tau}{2}}$$

Dual BGG resolution:

$$0 \to \mathcal{E}^{(\underline{k},w)} \to \omega^{(2-\underline{k},w)} \to \bigoplus_{\tau_0 \in \mathrm{Hom}(F,\mathbb{C})} \omega^{\left(\substack{2-k_\tau\ \tau\neq 0 \\ k_{\tau_0}},\ w\right)} \to \cdots \to \omega^{(\underline{k},w)} \to 0$$

Then $H^{[F:\mathbb{Q}]}_{\mathrm{Betti}}(X, \mathcal{E}^{(\underline{k},w)})$ is given by (when $k_\tau \geqslant 2$ & not all $k_\tau = 2$)

$$H^0(X, \omega^{(\underline{k},w)}) - \bigoplus_{\tau_0 \in \mathrm{Hom}(F,\mathbb{C})} H^1\left(X, \omega^{\left(\substack{k_\tau\ \tau\neq\tau_0 \\ 2-k_{\tau_0}},\ w\right)}\right) - \cdots - H^{[F:\mathbb{Q}]}(X, \omega^{(2-\underline{k},w)})$$

Proof of theorem: First consider the case when $\lambda = 0$

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & \underline{\mathbb{C}} \arrow[r] & \mathcal{O}_{\check{D}} \arrow[r, "d"] & \Omega^1_{\check{D}} \arrow[r, "d"] \arrow[d, phantom, "\parallel"] & \Omega^2_{\check{D}} \arrow[r] \arrow[d, phantom, "\parallel"] & \cdots \arrow[r] & \Omega^{\dim \check{D}}_{\check{D}} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & & & \underline{\mathfrak{p}^-} & \underline{\wedge^2 \mathfrak{p}^-} & & \underline{\wedge^{top} \mathfrak{p}^-} & \end{tikzcd}$$

$$\mathfrak{p}^+ = (\mathfrak{p}^-)^*$$

$\longleftrightarrow$ corresponds to

$$\cdots \to \underbrace{U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} \wedge^2 \mathfrak{p}^+}_{} \to \underbrace{U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} \mathfrak{p}^+}_{} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} \mathbb{C} \to \mathbb{C}$$

lowest wt $= \alpha_i$ for some $\alpha_i \in \Phi^-_{nc}$

lowest wt $= \alpha_i + \alpha_j$ for $\alpha_i \neq \alpha_j$ in Φ^-_{nc}

Tensoring $(*)$ with the loc. const sheaf $\underline{V}(\lambda)$, we get

$$0 \to \underline{V}(\lambda) \to \mathcal{O}_{\check{D}} \otimes \underline{V}(\lambda) \xrightarrow{d} \Omega^1_{\check{D}} \otimes \underline{V}(\lambda) \to \cdots$$

corresponding to

$$(**) \qquad \cdots \to \underbrace{\left(U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} \wedge^i(\mathfrak{p}^+)\right) \otimes V(\lambda)^*}_{} \to \cdots \to \left(U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} \mathbb{C}\right) \otimes V(\lambda)^* \to V(\lambda)^*$$

$$\cong \quad U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} \left(V(\lambda)^* \otimes \wedge^i(\mathfrak{p}^+)\right)$$

lowest wt λ

E.g.

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1} \to 0$$

$$0 \to \mathrm{Sym}^{k-2}\underline{\mathbb{C}}^2 \to \mathcal{O}_{\mathbb{P}^1} \otimes \mathrm{Sym}^{k-2}\underline{\mathbb{C}}^2 \to \Omega^1_{\mathbb{P}^1} \otimes \mathrm{Sym}^{k-2}\underline{\mathbb{C}}^2 \to 0$$

$$\longleftrightarrow \quad 0 \to \left(U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} (\alpha)\right) \otimes (\mathrm{Sym}^{k-2})^* \to \left(U(\mathfrak{g}_c) \otimes_{U(\mathfrak{q})} \mathbb{C}\right) \otimes (\mathrm{Sym}^{k-2})^* \to (\mathrm{Sym}^{k-2})^* \to 0$$

$$\cong \qquad\qquad \parallel$$

decreasing order

b/c as a module for opposite Borel, highest wt part is the sub. $\mathcal{U}(\mathfrak{g}_c) \otimes_{\mathcal{U}(\mathfrak{q})} \left(\frac{k}{2} \; \frac{k-2}{2} - \cdots \frac{4-k}{2}\right)$ $\qquad \mathcal{U}(\mathfrak{g}_c) \otimes_{\mathcal{U}(\mathfrak{q})} \left(\frac{k-2}{2} - \cdots \frac{2-k}{2}\right).$

<u>Key step</u>: Consider the action of $Z(\mathcal{U}(\mathfrak{g}))$, the only term with the same central character as $V(\lambda)^*$ are meaningful.

i.e. $(**) = \bigoplus$ complexes $\leftrightarrow$ corresponding to different generalized evals of $Z(\mathcal{U}(\mathfrak{g}))$

& hence, each summand is also exact.

<u>Fact</u>: In $\left(\mathcal{U}(\mathfrak{g}_c) \otimes_{\mathcal{U}(\mathfrak{q})} (V(\lambda)^* \otimes \wedge^i(\mathfrak{p}^+)\right)$, the terms that are left are

$$\bigoplus_{\substack{w \in W_n \\ \ell(w) = i}} \mathcal{U}(\mathfrak{g}_c) \otimes V\big(w(-\lambda-\rho)+\rho\big)$$

In our example, get $\mathcal{U}(\mathfrak{g}_c) \otimes_{\mathcal{U}(\mathfrak{q})} \left(\frac{k}{2}\right) \longrightarrow \mathcal{U}(\mathfrak{g}_c) \otimes_{\mathcal{U}(\mathfrak{q})} \left(\frac{2-k}{2}\right) \to (\mathrm{Sym}^{k-2})^* \to 0$

$$\rightsquigarrow \quad \underline{x}^{k-2} \longrightarrow \underline{x}^{-k}$$

$\rightsquigarrow$ dual back to sheaves: $0 \to \mathrm{Sym}^{k-2}(R^1\pi_* \underline{\mathbb{C}}) \to \omega^{2-k} \longrightarrow \omega^k \longrightarrow 0$