Automorphic Forms and Automorphic Vector Bundles

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Contents

Happy Dragon Boat Festival!

1 Automorphic Forms and Automorphic Representations

Denote $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_f = \mathbb{A}_{\mathbb{Q},f}$ be the finite adele.

Let G be a reductive group over $\mathbb{Q}$, Z is the center of G and $\mathfrak{g}$ is the complexification of the Lie algebra of $G(\mathbb{R})$, $U(\mathfrak{g})$ and $Z(\mathfrak{g})$ stands for the universal enveloping algebra and the center of universal enveloping algebra respectively.

$K_{\max,f} \subset G(\mathbb{A}_f)$ be a maximal compact subgroup, $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup of Lie group $G(\mathbb{R})$. Denote $K_{\max} = K_{\max,f} \times K_\infty$.

Definition 1.1 (Following [1]). A function $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is called an **automorphic form** if

- φ is smooth. i.e. locally constant on the non-archimedean part and C^∞ on the archimedean part.

- φ is left $G(\mathbb{Q})$ invariant.

- φ is $K_{\max}$ finite. i.e. The subspace generated by $K_{\max} \cdot \varphi$ is finite dimensional. Where the action is given by $(k\varphi)(g) = \varphi(gk)$.
- φ is $Z(\mathfrak{g})$ finite. i.e. φ is annihilated by a finite codimensional ideal of $Z(\mathfrak{g})$.
- φ is moderate growth.

The space of all automorphic forms is denoted by $\mathcal{A}(G)$.

Let $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a central character, define $\mathcal{A}(G;\omega)$ as follows

$$\mathcal{A}(G,\omega) = \{\varphi \in \mathcal{A}(G) | \varphi(zg) = \omega(z)\varphi(g), \forall z \in Z(\mathbb{A}), x \in G(\mathbb{A})\}$$

Remark. We will pretend that $G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{R})$ is compact, which is equivalent to say that there is no growth condition. Although, in our main example $G = \mathrm{GL}_2$ this will be false, but we pretend it is true(We will lost automorphic forms that are not cusp forms, for example, Eisenstein series)

Example 1.2. For $G = \mathrm{GL}_2, Z = \mathbb{G}_m$,in this case $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{C})$ and $Z(\mathfrak{g}) = \mathbb{C}[\Delta, Z]$, where $\Delta = EF + FE + \frac{1}{2}H^2$ is the Casimir operater and Z is the center of $\mathfrak{g}$.

Let $\Gamma = \Gamma_0(N)$. For a modular form $f \in S_k(\Gamma_0(N),\psi)$, we can define a adelic lift

$$\varphi_f(g) = f(g_\infty(\mathrm{i}))j(g_\infty,\mathrm{i})^{-k}\psi(k_0)$$

Which can be checked to be a automorphic form. Moreever, we have an isomorphism

$$S_k(\Gamma_0(N)) \cong \mathcal{A}_{\mathrm{cusp}}\left(\langle \Delta - \frac{1}{4}(k^2-1)\rangle, \sigma_k\right)^{\widehat{\Gamma_0(N)}}$$

(See [2])

Back to general discussion, We have a natural action of $G(\mathbb{A})$ on $\mathcal{A}(G)$ or $\mathcal{A}(G,\omega)$. i.e. these space is natural a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ module. The action is given by

$$(X, k, g, \varphi) \mapsto (g \mapsto X\varphi(xkg))$$

Recall that we have assumed that all automorphic forms are cuspidal, then we have a spectrum decomposition(called the **cuspidal spectrum**)

$$\mathcal{A}(G,\omega) = \bigoplus_\pi (\pi_\infty \otimes (\otimes_l' \pi_l))^{m(\pi)}$$

Where $m(\pi)$ is a finite number, called the **automorphic multiplicity**. $\otimes'$ is the restricted tensor product. And for all but finitely many finite prime l,π_l is **unramified principal series**. and v_l^0 is the sperical vector for π_l.

(For a proof, see [1] Chapter 9 or [3] Chapter 3, where $\mathcal{A}$ is replaced by $\mathcal{A}_{\text{cusp}}$)

Here unramified principal series means $\pi_l = \operatorname{Ind}_{B(\mathbb{Q}_l)}^{G(\mathbb{Q}_l)}\chi_l$. Where B is Borel subgroup and T is the Levi quotient. And

$$\chi_l : T(\mathbb{Q}_l) \to T(\mathbb{Q}_l)/T(\mathbb{Z}_l) \to \mathbb{C}^\times$$

is an unramified character. As $G(\mathbb{Q}_l) = B(\mathbb{Q}_l)G(\mathbb{Z}_l)$ (Iwasawa decomposition), so if $\varphi \in \left(\operatorname{Ind}_{B(\mathbb{Q}_l)}^{G(\mathbb{Q}_l)}\chi_l\right)^{G(\mathbb{Z}_l)}$. Then

$$\varphi(g) = \varphi(bk) = \chi_l(b)\varphi(1)$$

So $\dim \pi_l^{G(\mathbb{Z}_l)} = 1$, we then choose $v_l^0 = \varphi(1)$

Example 1.3. For $G = \mathrm{GL}_2, B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

χ_l is of the form $\begin{pmatrix} \mathbb{Q}_l^\times & 0 \\ 0 & \mathbb{Q}_l^\times \end{pmatrix} \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}, (x, y) \mapsto \alpha^{v_l(x)}\beta^{v_l(y)}$

Definition 1.4. An **automorphic representation** is a representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ of the form $\pi_\infty \otimes (\otimes_l' \pi_l)$ which occurs in the cuspidal spectrum decomposition(for some ω).

Remark. In general, an **automorphic representation** is an irreducible admissible subrepresentation of natural action of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ on $\mathcal{A}(G)$. And automorphic representation defined above is called **cuspidal automorphic representation**.

2 Automorphic Vector Bundles and Automorphic Representations

Fix an algebraic representation W of K_∞. And fix an open compact subgroup $K_f \subset G(\mathbb{A}_f)$.

We have an locally symmetric space(real manifolds)

$$\mathrm{Sh}_G(K_f) = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}_f)/K_f\right) \times \left(G(\mathbb{R})/K_\infty\right)$$

and an vector bundle on $\mathrm{Sh}_G(K_f)$:

$$\underline{W} = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}_f)/K_f\right) \times \left(G(\mathbb{R}) \times^{K_\infty} W\right)$$

So we have

$$\{C^\infty \text{ sections of } \underline{W}\} = \{\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to W, K_\infty \text{ equivariant, } K_f \text{ invariant}\}$$

Which is roughly equal to $(\mathcal{A}(G, \omega)^{K_f} \otimes W)^{K_\infty}$ if we omit the cusp issue and choose ω compatible with W.

Example 2.1. For $G = \mathrm{GL}_2$

List of irreducible admissible $(\mathfrak{g}, K)$ representation π_∞: ([4], Chapter 7 or [3] 2.3):

- **holomorphic discrete series**:

$$\pi_\infty = \pi_k \oplus \pi_{k+2} \oplus \cdots$$

  Where $k > 0$, π_l stands for $K = \mathrm{SO}(2)$ acts as $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto \mathrm{e}^{2\pi \mathrm{i} l\theta}$. And the $\mathfrak{g}$ action is like an "anti" Verma module with integer weights:

$$Ev_l = (\frac{k}{2} + l)v, Fv_l = (\frac{k}{2} - l)v_{l-2}$$

- **antiholomorphic discrete series**:

$$\pi_\infty = \cdots \pi_{-k-4} \oplus \pi_{-k-2} \oplus \pi_{-k}$$

  where $k > 0$ and $\mathfrak{g}$ acts as a Verma module of integral weight.

- **principal series**

$$\pi_\infty = \cdots \pi_{k-2} \oplus \pi_k \oplus \pi_{k+2} \oplus \cdots$$

  $\mathfrak{g}$ action looks like "Verma module with non-integer weight"

- finite dimensional representation

- **limit of discrete series**(corresponds to weight 1 form)

Fix the representation

$$\chi_{-k} : K_\infty = \mathbb{R}^\times \cdot \mathrm{SO}(2) \to \mathbb{C}^\times, (r, z) \mapsto z^{-k}$$

From previous lecture, we learned that this corresponds to ω_k

And we have

$$C^\infty(\mathrm{Sh}_G(K_f), \underline{W}_{-k}) \doteq (\mathcal{A}(G)^{K_f} \otimes W_{-k})^{K_\infty} = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes (\pi_\infty \otimes W_{-k})^{K_\infty}$$

The last term is just $\pi_\infty^{K_\infty = \chi_k}$

It has a subspace

$$\mathrm{Hol}(\mathrm{Sh}_G(K_f), \underline{W}_{-k}) = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes (\pi_\infty^{K_\infty = \chi_k})^{F=0}$$

So k is the lowest weight, by the list of classification above, when $\pi_\infty \otimes W_{-k})^{K_\infty}$ non zero, π_∞ must be discrete series representation.

We can use above discussion to explain old/new form theory.

$$S_k(\Gamma) = \mathrm{Hol}(\mathrm{Sh}_G(\widehat{\Gamma}), \chi_{-k}) = \bigoplus_{\underline{\pi}} (\pi_f^{\hat{\Gamma}}) \otimes \pi_\infty^{K_\infty = \chi_k}$$

Consider $\Gamma = \Gamma_1(N), p \not| N, K = \widehat{\Gamma} = K^{(p)}\mathrm{GL}_2(\mathbb{Z}_p)$. And $\Gamma \supset \Gamma' = \Gamma_1(N) \cap \Gamma_0(p), K' = K^{(p)}\mathrm{Iw}_p$.Where

$$\mathrm{Iw}_p = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

is the Iwasawa subgroup.

We have two types of operators, which fits into the following diagram

$$S_k(\Gamma) \quad = \quad \bigoplus\nolimits_\pi (\pi_f^{(p),K^{(p)}})^{\oplus m(\pi)} \otimes (\pi_p)^{\mathrm{GL}_2(\mathbb{Z}_p)}$$

$$S_k(\Gamma') \quad = \quad \bigoplus\nolimits_\pi (\pi_f^{(p),K^{(p)}})^{\oplus m(\pi)} \otimes (\pi_p)^{\mathrm{Iw}_p}$$

New forms: $= \bigoplus_\pi (\pi_f^{(p),K^(p)})^{\oplus m(\pi)} \otimes (\pi_p)^{\mathrm{Iw}_p}$,sum for those π such that $\pi_p^{\mathrm{GL}_2(\mathbb{Z}_p)} = 0$ but $\pi_p^{\mathrm{Iw}_p} \neq 0$. Special for GL_2 theory, for these π, $\dim \pi_p^{\mathrm{Iw}_p} = 1$.

Old Forms: If π_p is an unramified principal series, we have an isomorphism

$$(\pi_p)^{\mathrm{GL}_2(\mathbb{Z}_p), \oplus 2} \cong (\pi_p)^{\mathrm{Iw}_p}$$

3 Relations to $(\mathfrak{g}, K)$ Cohomology

Now, we move to the general situation:

Recall that we have gotten

$$\{C^\infty \text{ sections of } \underline{W}\} = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes (\pi_\infty \otimes W)^{K_\infty}$$

Let's assume that we are in the situation of Shimura varieties, we want cohomology for holomorphic sections $H^\bullet(\mathrm{Sh}_G(K_f), \underline{W})$.

Use Dolbeault resolution

$$\mathcal{O}^{\mathrm{hol}}_{\mathrm{Sh}_G(K_f)} \longrightarrow \underline{C^\infty} \xrightarrow{\bar\partial} \overline{\Omega}^1 \xrightarrow{\bar\partial} \overline{\Omega}^2 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \overline{\Omega}^d \longrightarrow 0$$

Note that $T_{\mathbb{C}}\mathrm{Sh}_G(K_f) = \mathfrak{g}/\mathfrak{q} = \mathfrak{p}^+$,so $\Omega^1 \cong \underline{(\mathfrak{p}^+)^*}$ and $\overline{\Omega}^1 \cong \underline{(\mathfrak{p}^-)^*}$

Tensoring with $\underline{W}$, we get a resolution of $\underline{W}^{\mathrm{hol}}$, which implies

$$H^\bullet(\mathrm{Sh}_G(K_f), \underline{W}^{\mathrm{hol}}) = H^\bullet\left(\bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes (\pi_\infty \otimes \mathrm{Hom}(\wedge^\bullet(\mathfrak{p}^-), W))\right)$$

And the latter term is exactly the $(\mathfrak{g}, K)$ cohomology we now introduce.

What is $(\mathfrak{g}, K)$ cohomology?[5]

(1) Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$ module. Define

$$C^q(\mathfrak{g}; V) = \mathrm{Hom}_k \Lambda^q(\mathfrak{g}), V)$$

and $\mathrm{d} : C^q(\mathfrak{g}; V) \to C^{q+1}(\mathfrak{g}, V)$ is given by

$$\mathrm{d}f(x_0, \cdots, x_q) = \sum_i (-1)^i x_i f(x_0, \cdots, \widehat{x_i}, \cdots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \cdots, \widehat{x_i}, \cdots, \widehat{x_j}, \cdots, x_q)$$

The cohomology is denoted by $H^\bullet(\mathfrak{g}, V)$. Which is also the right derived functor of $V \mapsto V^{\mathfrak{g}} := \{v \in V | xv = 0, \forall x \in \mathfrak{g}\}$.i.e.

$$H^p(\mathfrak{g}, V) = \mathrm{Ext}^p_{U\mathfrak{g}}(k, V)$$

where k is the base field.

(2) Relative Lie algebra cohomology

Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra.Define

$$C^q(\mathfrak{g}, \mathfrak{k}, V) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V)$$
$$= \{f \in C^q(\mathfrak{g}, V) | f \text{ only depends on } x_i \in \mathfrak{g}/\mathfrak{k}, \sum_i f(x_1, \cdots, [x, x_i], \cdots, x_q) = xf(x_1, \cdots, x_q), \text{ for } x \in \mathfrak{k}\}$$

It can be verified that $C^\bullet(\mathfrak{g}, \mathfrak{k}, V)$ is a sub-cochain complex of $C^\bullet(\mathfrak{g}, V)$, whose cohomology is denoted by $H^\bullet(\mathfrak{g}, \mathfrak{k}, V)$.

(3) $(\mathfrak{g}, K)$ cohomology and $(\mathfrak{q}, K)$ cohomology.

Now we work over $\mathbb{C}$, let $\mathfrak{g}$ be a Lie algebra, not necessrily reductive.

Let G be a real Lie group with complexified Lie algebra $\mathfrak{g}$ and K be the maximal compact with connected component K^0.

Let V be a $(\mathfrak{g}, K)$ module, define

$$C^q(\mathfrak{g}, K, V) := \mathrm{Hom}_K(\Lambda^q(\mathfrak{g}/\mathfrak{k}), V) \cong C^q(\mathfrak{g}, \mathfrak{k}, V)^{K/K^0}$$

Whose cohomology groups is denoted by $H^\bullet(\mathfrak{g}, K; V)$.

Proposition 3.1 (See [5]).

$$H^q(\mathfrak{g}, K; V) = H^q(\mathfrak{g}, \mathfrak{k}; V)^{K/K^0} = \mathrm{Ext}^q_{\mathfrak{g},K}(\mathbb{C}, V)$$

Theorem 3.2.

$$H^\bullet(\mathrm{Sh}_G(K_f), \underline{W}) = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes H^\bullet(\mathfrak{q}, K; \pi_\infty \otimes W)$$

A deep theorem

Theorem 3.3. When π_∞ is a discrete series or limit of discrete series, W is irreducible.Then $H^q(\mathfrak{q}, K_\infty; \pi_\infty \otimes W)$ is non zero at exactly one degree if $G(\mathbb{R})$ is connected. And in this case the dimension of the non-vanishing cohomology group is 1.

Example 3.4. For $G = \mathrm{GL}_2$,π_∞ is the discrete series $\pi_k \oplus \pi_{k+2} \oplus \cdots$

We have

$$\dim H^0(\mathfrak{q}, K_\infty; \pi_\infty \otimes W_{-k}) = 1$$

and

$$\dim H^1(\mathfrak{q}, K_\infty; \overline{\pi}_\infty \otimes W_{k-2}) = 1$$

Example 3.5. Let F be a totally real field, $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGL}_2$ weight $\underline{k} = (k_\tau)_{\tau \in \mathrm{Hom}(F,\mathbb{R})}$, all are even. $n_0 = \#\{\tau \in \mathrm{Hom}(F, \mathbb{R}), k_\tau \leq 0\}$

By the multiplicity one theorem for PGL_n

$$H^\bullet(\mathrm{Sh}_G(K_f), \omega^{k_\tau}) = \bigoplus_\pi \left((\pi_f)^{K_f} \otimes \bigotimes_{\tau \in \mathrm{Hom}(F,\mathbb{R})} H^\bullet(\mathfrak{q}_\tau, K_{\infty,\tau}; \pi_\tau \otimes \chi_{-k_\tau}) \right)$$

And $H^\bullet(\mathfrak{q}_\tau, K_{\infty,\tau}; \pi_\tau \otimes \chi_{-k_\tau})$ is concentrated in one degree and dim 1. It concentrates in degree 0 if $k_\tau \geq 2$ and in degree 1 if $k_\tau \leq 0$.

So $H^\bullet(\mathrm{Sh}_G(K_f), \omega^{k_\tau})$ is concentrated in degree n_0.

Another Relationship between cohomology of vector bundles and $(\mathfrak{g}, K)$ cohomology When V is an algebraic $\mathbb{C}$ representation of G defined over a number field. We can associated a locally constant sheaf $\underline{V}$ on $\mathrm{Sh}_G(K_f)$. $\mathcal{V} = \underline{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Sh}_G(K_f)}$ is the deRham local system.

We get

$$\begin{aligned}
H^\bullet_{\mathrm{Betti}}(\mathrm{Sh}_G(K_f), \underline{V}) &= \mathbb{H}^\bullet\left(\mathrm{Sh}_G(K_f), \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathrm{Sh}_G(K_f)} \to \cdots \Omega^d_{\mathrm{Sh}_G(K_f)} \to \cdots\right) \\
&= \mathbb{H}^\bullet(\mathrm{Sh}_G(K_f), \mathcal{V} \otimes \Omega^{\bullet,\bullet}_{\mathrm{Sh}_G(K_f)}) \\
&= \bigoplus (\pi_f^{K_f})^{\oplus m(\pi)} \otimes H^\bullet\left(\pi_\infty \otimes \mathrm{Hom}(\Lambda^\bullet \mathfrak{p}^+ \otimes \Lambda^\bullet \mathfrak{p}^-, V)^{K_\infty}\right) \\
&= \bigoplus (\pi_f^{K_f})^{\oplus m(\pi)} \otimes H^\bullet\left(\mathfrak{g}, K_\infty; \pi_\infty \otimes V\right)
\end{aligned}$$

Example 3.6. F totally real,$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGL}_{2,F}$

Langlands observation:

$$\dim H^{\mathrm{mid}}\left(\mathfrak{g}, K_\infty; \pi_\infty \otimes V(\lambda)\right) = \dim(\text{representation of } \widehat{G} \text{ of highest weight } \mu)$$

or rather this is how Langlands discovered the dual group.

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[5] Wallach Borel. *Continuous cohomology, discrete subgroups, and representations of reductive groups.*