Automorphic representations and cohomology of automorphic vector bundles

Main context $\qquad$ Special case of GL_2 or $\mathrm{Res}_{F/\mathbb{Q}} GL_2$ (F totally real) $\qquad$ General comments

Important comments

$*$ Let $G = GL_2$ be a reductive group over $\mathbb{Q}$

Let $Z = \mathbb{G}_m$ denote the center of G

Fix a unitary character $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \longrightarrow S^1 \subseteq \mathbb{C}^\times$

(This means $\mathbb{Q}^\times\backslash\mathbb{A}^\times \simeq \hat{\mathbb{Z}}^\times \times \mathbb{R}^\times_{>0} \longrightarrow S^1$)

$x \longmapsto e^{2\pi i s x}$ for some $s \in \mathbb{R}$

Fix $K_\infty = \mathbb{R}^\times \cdot SO(2)$ a max'l compact subgroup of $G(\mathbb{R})$

$A(G;\omega) := \{$ smooth functions $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$

s.t. (1) $\varphi(zx) = \omega(z)\varphi(x) \quad \forall z \in Z(\mathbb{A})$ & $x \in G(\mathbb{A})$

(2) $\exists$ an open compact subgroup $K \subseteq G(\mathbb{A}_f)$

s.t. $\varphi(xu) = \varphi(x) \quad \forall u \in K$.

$(k_\infty\varphi)(g) := \varphi(gk_\infty)$

(3) φ is K_∞-finite, i.e. the subspace generated by $K_\infty \cdot \varphi$ is fin. dim'l.

(4) For $Z(\mathfrak{g})$ = center of universal enveloping algebra of $\mathfrak{g}$

$= \mathbb{C}[$Casimir operator, central derivation$]$

Casimir operator $= EF + FE + \frac{1}{2}H^2$

φ is $Z(\mathfrak{g})$-finite, i.e. the subspace gen. by $Z(\mathfrak{g})\cdot\varphi$ is fin. dim'l.

(5) Growth condition at cusps $\}$

Will pretend that $G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{R})$ is compact today, so no cusp.

This will exclude $G = GL_2$, but we pretend this is okay; will be missing Eisenstein series

Then $A(G,\omega) \circlearrowleft G(\mathbb{A})$

$\oplus m(\pi)$ ← automorphic multiplicity

$\|$

$$\bigoplus_{\pi} \; \underset{\substack{\circlearrowleft \\ G(\mathbb{R})}}{\pi_\infty} \otimes \Big(\underset{\substack{\circlearrowleft \\ G(\mathbb{Q}_\ell)}}{\bigotimes_\ell{}' \, \pi_\ell} \Big) \qquad \text{spectral decomposition}$$

$\otimes'$ is the "restricted tensor product"

i.e. for each π, all but finitely many ℓ, π_ℓ is an "unramified principal series"*

let $v_\ell^\circ :=$ "spherical vector" in π_ℓ.

Then $\bigotimes'_\ell \pi_\ell$ is spanned by tensors $\bigotimes_\ell v_\ell$ where $v_\ell = v_\ell^\circ$ for all but finitely many ℓ.

* Here "unramified principal series" $\pi_\ell = \mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)} \chi_\ell$

for a Borel subgp B (with quotient T) $\qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \twoheadrightarrow T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

$$\chi_\ell : T(\mathbb{Q}_\ell) \twoheadrightarrow T(\mathbb{Q}_\ell)/T(\mathbb{Z}_\ell) \to \mathbb{C}^\times \quad \text{a character}$$

$$\begin{aligned} T(\mathbb{Q}_\ell) = \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times &\xrightarrow{v_\ell,\, v_\ell} \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C}^\times \\ (x,\, y) &\longmapsto \alpha^{v_\ell(x)} \cdot \beta^{v_\ell(y)} \end{aligned}$$

As $G(\mathbb{Q}_\ell) = B(\mathbb{Q}_\ell) \cdot G(\mathbb{Z}_\ell)$ Iwasawa decomposition

So if $\varphi \in \left(\mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)} \chi_\ell\right)^{G(\mathbb{Z}_\ell)}$

$$\Rightarrow \varphi(g) = \varphi(bk) = \chi_\ell(b) \cdot \varphi(1) \quad \text{for } g = bk \in B(\mathbb{Q}_\ell) G(\mathbb{Z}_p)$$

$$\Rightarrow \quad \pi_\ell^{G(\mathbb{Z}_\ell)} \simeq \mathbb{C} \qquad \leftarrow \text{also need } \chi \text{ to be trivial on } B(\mathbb{Q}_\ell) \cap G(\mathbb{Z}_p) = B(\mathbb{Z}_p).$$

$$\varphi \longmapsto \varphi(1)$$

spherical vector. $v_\ell^\circ \longleftarrow 1$

- <u>Definition</u>. An <u>automorphic representation</u> is an rep'n $\pi = \pi_\infty \otimes \bigotimes'_\ell \pi_\ell$ of $G(\mathbb{A})$ that appear in $\mathcal{A}(G;\omega)$ for some ω.

- Fix an algebraic rep'n W of K_∞

  e.g. $\chi_k : K_\infty = \mathbb{R}^\times \cdot SO(2) \longrightarrow \mathbb{C}^\times$ $\qquad \longleftrightarrow$ by previous lecture, this corresponds to ω^k

  $$(r,\, z) \longmapsto z^{-k}.$$

Fix an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$.

$$\underline{W} = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}_f)/K_f \times \left(G(\mathbb{R}) \overset{K_\infty}{\times} W \right) \right) \longleftarrow \text{vector bundle assoc. to } W$$

$|$

$$\rightsquigarrow Sh_G(K_f) = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}_f)/K_f \times G(\mathbb{R})/K_\infty \right)$$ ← just a locally symmetric space. real manifold.

$$\Rightarrow C_{cont}\left(Sh_G(K_f), \underline{W}\right) = \{ C^\infty\text{- sections of } \underline{W} \}$$

$$= \{ \varphi : G(\mathbb{Q}) \backslash G(\mathbb{A}) \xrightarrow{K_\infty \circlearrowleft} W \}$$

$$\doteq \left(\mathcal{A}(G;\omega)^{K_f} \otimes W \right)^{K_\infty}.$$

Case of GL_2: $C_{cont}(Sh_G(K_f), \underline{W}_{-k}) \overset{\bullet}{=} \left(\mathcal{A}(G;\omega)^{K_f} \otimes \underline{W}_{-k} \right)^{K_\infty}$ ← cusp issue

$$= \bigoplus_{\pi} \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes \underbrace{\left(\pi_\infty \otimes \underline{W}_{-k}\right)^{K_\infty}}_{\pi_\infty^{K_\infty = \chi_k}}$$ ← 1-dim'l

$$Hol\left(Sh_G(K_f), \underline{W}_k\right) = \bigoplus_{\pi} \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes \left(\left(\pi_\infty^{K_\infty = \chi_k}\right)^{F=0} \right)$$

↑ lowest weight vectors in π_∞

Comes from here.

See e.g. Goldfeld-Hundley §7.

In this case there are several possibilities of π_∞: (confusing GL_2 w/ SL_2)

(1) holomorphic discrete series:

$$\pi_\infty = \pi_k \oplus \pi_{k+2} \oplus \pi_{k+4} \cdots\cdots \qquad k>0$$

K_∞ acts via χ_k, χ_{k+2}, χ_{k+4}; $\chi_k \xrightarrow{E} \chi_{k+2} \xrightarrow{E} \chi_{k+4}$, $\chi_{k+4} \xrightarrow{F} \chi_{k+2} \xrightarrow{F} \chi_k$

$\circlearrowleft$ $\mathfrak{g}$-action looks like a "anti" Verma module with integer weights

(2) antiholomorphic discrete series:

$$\pi_\infty = \cdots\cdots \pi_{-k-4} \oplus \pi_{-k-2} \oplus \pi_{-k} \qquad k>0$$

χ_{-k-4}, χ_{-k-2}, χ_{-k}; $\chi_{-k-4} \xrightarrow{E} \chi_{-k-2} \xrightarrow{E} \chi_{-k}$

$\circlearrowleft$ $\mathfrak{g}$-action looks like a Verma module

(3) principal series $\pi_\infty = \cdots \oplus \pi_{k-2} \oplus \pi_k \oplus \pi_{k+2} \oplus \cdots$ infinite both ways,

$\mathfrak{g}$-action looks like "Verma module with non-integer weights"

(4) finite dim'l rep'n. (doesn't appear here)

(5) limit of discrete series $\leftrightarrow$ wt 1 forms

Explain old/new form theory in GL_2:

Consider only those automorphic rep'ns π whose π_∞ is a discrete series of wt k.

$$S_k(\Gamma) \cong \mathrm{Hol}(Sh_G(\hat{\Gamma}), \underline{\chi}_k) = \bigoplus_\pi (\pi_f^{\hat{\Gamma}})^{\oplus m(\pi)} \otimes \pi_\infty^{K_\infty = \chi_k}$$

always one-dim'l, ignore ...

Consider $\Gamma = \Gamma_1(N)$ $\quad p \nmid N \rightsquigarrow K = \hat{\Gamma} = K^{(p)} GL_2(\mathbb{Z}_p)$

$\Gamma' = \Gamma_1(N) \cap \Gamma_0(p) \subseteq \Gamma$ $\quad K' = K^{(p)} \cdot Iw_p$, $\quad Iw_p = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$

Then:

$$\begin{array}{ccc} S_k(\Gamma) & = & \bigoplus_\pi (\pi_f^{(p),K^{(p)}})^{\oplus m(\pi)} \otimes (\pi_p)^{GL_2(\mathbb{Z}_p)} \\ f(z)\mapsto f(z) \downarrow\downarrow f(z)\mapsto f(pz) & & \downarrow\downarrow \; v\mapsto v, \; v \mapsto \begin{pmatrix} p & \\ & 1\end{pmatrix} v \\ S_k(\Gamma') & = & \bigoplus_\pi (\pi_f^{(p)K^{(p)}})^{\oplus m(\pi)} \otimes (\pi_p)^{Iw_p} \end{array}$$

$$\text{New forms} = \bigoplus_\pi (\pi_f^{(p)K^{(p)}})^{\oplus m(\pi)} \otimes (\pi_p)^{Iw_p}$$

those π s.t. $\pi_p^{GL_2(\mathbb{Z}_p)} = 0$ but $\pi_p^{Iw_p} \neq 0$

special for GL_2-theory: for these π_p, $\pi_p^{Iw_p} = $ 1-dim

Old form: If π_p is an unramified principal series,

$$(\pi_p)^{GL_2(\mathbb{Z}_p), \oplus 2} \xrightarrow[v\mapsto \begin{pmatrix} p & \\ & 1\end{pmatrix} v]{v \mapsto v} \underbrace{(\pi_p)^{Iw_p}}$$ is an isom. unless T_p-eigenvalue is strange.

2-dim'l b/c $\# B(\mathbb{Q}_p)\backslash GL_2(\mathbb{Q}_p)/Iw_p = 2$

(In general, $B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/Iw_p \simeq$ Weyl gp)

Now, move to the general situation:

Recall:
$$C_{cont}(Sh_G(K_f), \underline{W}) \cong \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes (\pi_\infty \otimes W)^{K_\infty}$$

Let's assume that we are in the situation of Shimura varieties

We want cohomology for holomorphic sections $H^*(Sh_G(K_f), \underline{W})$

Use resolution $0 \to \mathcal{O}^{hol}_{Sh_G(K_f)} \to \underline{C}^\infty \xrightarrow{\bar\partial} \underline{\bar\Omega}^1 \xrightarrow{\bar\partial} \underline{\bar\Omega}^2 \to \cdots \to \underline{\bar\Omega}^{\dim Sh_G(K_f)} \to 0$

$$\underline{\bar\Omega}^1 = \underline{C}^\infty \otimes (\underline{\mathfrak{p}}^-)^*, \quad \underline{\bar\Omega}^2 = \underline{C}^\infty \otimes \wedge^2 (\underline{\mathfrak{p}}^-)^*$$

b/c we've learned Tangent $= \underline{\mathfrak{g}/\mathfrak{p}^-} \Rightarrow \Omega^1 \cong \underline{\mathfrak{p}}^-$

$\rightsquigarrow$ tensor with $\underline{W}$ $\to$ resolution of $\underline{W}^{hol}$

$$\Rightarrow H^*(Sh_G(K_f), \underline{W}) = H^*\left(\bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes \left(\pi_\infty \otimes \mathrm{Hom}(\wedge^\bullet(\mathfrak{p}^-), W)\right)^{K_\infty} \right)$$

$$=: \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes \underbrace{H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes W)}_{(\mathfrak{g}, K_\infty)\text{-cohomology.}}$$

* <u>What is $(\mathfrak{g}, K_\infty)$-cohomology?</u> (E.g. [Borel-Wallach, Chap 1])

1. Lie algebra cohomology

   $\mathfrak{g}$ Lie algebra $\curvearrowright V$ vector space.

   Define $C^q(\mathfrak{g}; V) := \mathrm{Hom}(\wedge^q(\mathfrak{g}), V)$

   $d: C^q(\mathfrak{g}; V) \to C^{q+1}(\mathfrak{g}; V)$ is given by

   $$df(x_0, \dots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \dots, \hat{x}_i, \dots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_q)$$

   The cohomology is $H^*(\mathfrak{g}; V)$ with $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}=0}$.

2. Relative Lie algebra cohomology

   $\mathfrak{k} \subseteq \mathfrak{g}$ Lie subalgebra $\curvearrowright V$ vector space

   Define $C^q(\mathfrak{g}, \mathfrak{k}; V) := \mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), V) \hookrightarrow C^q(\mathfrak{g}; V)$

   $$= \left\{ f: \mathfrak{g} \to V;\ \text{s.t.}\ \begin{array}{l} f(x_1, \dots, x_q) \text{ depends only on each } x_i \in \mathfrak{g}/\mathfrak{k} \\ \sum_i f(x_1, \dots, [x, x_i], \dots, x_q) = x \cdot f(x_1, \dots, x_q) \text{ for } x \in \mathfrak{k} \end{array} \right\}$$

   Can show that d sends $C^q(\mathfrak{g}, \mathfrak{k}; V)$ into $C^{q+1}(\mathfrak{g}, \mathfrak{k}; V)$

   $\rightsquigarrow$ The cohomology is $H^*(\mathfrak{g}, \mathfrak{k}; V)$.

③ $(\mathfrak{g},K)$-cohomology and $(\mathfrak{q},K)$-cohomology.

Let $\mathfrak{g}$ be a Lie alg, not necessarily reductive (so either $\mathfrak{g}$ or $\mathfrak{q}$).

$K :=$ max'l compact subgroup $(K = K_\infty) \supseteq K^\circ =$ connected component of K

Assume that K is reductive.

Define $C^q(\mathfrak{g},K;V) := \mathrm{Hom}_K(\Lambda^q(\mathfrak{g}/\mathfrak{k}),V) \cong C^q(\mathfrak{g},\mathfrak{k};V)^{K/K^\circ}$

So we have $H^*(\mathfrak{g},K;V) := H^*(\mathfrak{g},\mathfrak{k};V)^{K/K^\circ}$

__Theorem.__ $H^*(Sh_G(K_f),\underline{W}) = \bigoplus_\pi (\pi_f^{K_f})^{\oplus m(\pi)} \otimes H^*(\mathfrak{q},K_\infty;\pi_\infty\otimes W)$

__Deep Theorem__ When π_∞ is a discrete series or limit of discrete series, W irred.

$H^*(\mathfrak{q},K_\infty;\pi_\infty\otimes W)$ is nonzero at exactly one degree, if $G(\mathbb{R})$ is connected.

& in this case, dim $=1$.

__Example__ $G = GL_2$, π_∞ discrete series $\pi_k \oplus \pi_{k+2} \oplus \cdots$

$$\rightsquigarrow \begin{cases} H^0(\mathfrak{q},K_\infty;\pi_\infty\otimes W_{-k}) = 1\text{-dim'l} \\ H^1(\mathfrak{q},K_\infty;\underbrace{\overline{\pi}_\infty\otimes W_{k-2}}) = 1\text{-dim'l} \end{cases}$$

$$\shortparallel$$

$$\Big(\cdots\cdots \oplus \underbrace{\overline{\pi}_{k+2}\otimes\chi_{k-2}}_{wt=-4} \oplus \underbrace{\overline{\pi}_k\otimes\chi_{k-2}}_{wt=-2}\Big) \otimes \Big(\mathbb{C} \longrightarrow \overset{\substack{wt=2\\ \downarrow}}{\mathfrak{p}^+}\Big)$$

__Example__: $F =$ totally real field. $G = \mathrm{Res}_{F/\mathbb{Q}} PGL_2$

weight $\underline{k} = (k_\tau)_{\tau\in\mathrm{Hom}(F,\mathbb{R})}$, k_τ all even.

$n_0 := \#\{\tau\in\mathrm{Hom}(F,\mathbb{R})\ ;\ k_\tau \leq 0\}$

$$H^*(Sh_G(K_f),\omega^{k_\tau}) \underset{\uparrow}{=} \bigoplus_\pi (\pi_f)^{K_f} \otimes \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{R})} \underbrace{H^*(\mathfrak{q}_\tau,K_{\infty,\tau};\pi_\tau\otimes\chi_{-k_\tau})}_{\uparrow}$$

multiplicity one holds for PGL_n

Concentrated in one degree & dim 1.
in degree 0 if $k_\tau \geq 2$
in degree 1 if $k_\tau \leq 0$.

So $H^*(Sh_G(K_f), \omega^{k_T})$ is concentrated in degree n_0

When V is an algebraic $\mathbb{C}$-rep'n of G (def'd over a number field)

$\rightsquigarrow$ $\underline{V}$ = locally constant sheaf assoc. to V

$$\begin{array}{c} \underline{V} \\ \downarrow \\ Sh_G(K_f) \end{array} \qquad \bullet \ \mathcal{V} := \underline{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_{Sh_G(K_f)} \leftarrow \text{de Rham local system}$$

$$(1 \otimes \nabla_{Sh_G(K_f)})$$

Get $H^*_{\text{Betti}}\left(Sh_G(K_f)(\mathbb{C}), \underline{V}\right) = \mathbb{H}^*\left(Sh_G(K_f), \mathcal{V} \xrightarrow{1\otimes\nabla} \mathcal{V}\otimes\Omega^1_{Sh_G(K_f)} \to \cdots \to \mathcal{V}\otimes\Omega^d_{Sh_G(K_f)}\right)$

$$= \mathbb{H}^*\Big(Sh_G(K_f), \underbrace{\mathcal{V}\otimes\Omega^{\cdot,\cdot}_{Sh_G(K_f)}}_{C^\infty\text{-resolution}}\Big)$$

$$= \bigoplus_{\pi} \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes H^*\left(\left(\pi_\infty \otimes \mathrm{Hom}\left(\wedge^{\cdot}(\mathfrak{p}^+)\otimes\wedge^{\cdot}(\mathfrak{p}^-), V\right)\right)^{K_\infty}\right)$$

$$= \bigoplus_{\pi} \left(\pi_f^{K_f}\right)^{\oplus m(\pi)} \otimes H^*\left(\mathfrak{g}, K_\infty; \pi_\infty \otimes V\right)$$

<u>Example</u>: F totally real, $G = \mathrm{Res}_{F/\mathbb{Q}} GL_{2,F}$,

Weight $\left((k_\tau)_{\tau\in\mathrm{Hom}(F,\mathbb{R})}, w\right) \rightsquigarrow k_\tau \equiv w \bmod 2$, $k_\tau \geq 2$.

$$\begin{array}{c} A \\ \downarrow \pi \\ Sh_G(K_f)_{/\mathbb{C}} \end{array} \qquad (R^1\pi_*\underline{\mathbb{C}})^* = \bigoplus_{\tau\in\mathrm{Hom}(F,\mathbb{R})} \mathcal{H}_{1,\tau}$$

$$\mathcal{H}_{1,\tau} \otimes \mathcal{O}_{Sh_G(K_f)} \simeq \mathcal{H}_1^{dR}\left(A/Sh_G(K_f)\right)_\tau =: \mathcal{H}^{dR}_{1,\tau}$$

Get $\mathcal{H}^{(\underline{k},w)} := \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{R})} \mathrm{Sym}^{k_\tau-2}\mathcal{H}_{1,\tau} \otimes \left(\wedge^2\mathcal{H}_{1,\tau}\right)^{\otimes\frac{w-k_\tau}{2}}$

$$\mathcal{H}^{(\underline{k},w)}_{dR} \cong \bigotimes_{\tau\in\mathrm{Hom}(F,\mathbb{R})} \mathrm{Sym}^{k_\tau-2}\mathcal{H}^{dR}_{1,\tau} \otimes \left(\wedge^2\mathcal{H}^{dR}_{1,\tau}\right)^{\otimes\frac{w-k_\tau}{2}}$$

$$H^*\left(Sh_G(K_f), \mathcal{H}^{(\underline{k},w)}\right) \simeq H^*\left(Sh_G(K_f), \mathcal{H}^{(\underline{k},w)} \otimes \Omega^{\bullet} \ldots\right)$$

$$H^1_{\mathrm{Betti}}\left(Sh_G(K_f), \ldots\right)_{\mathrm{cusp}} = H^1\left(Sh_G(K_f), \ldots_{dR}\right) \otimes S\left(Sh_G(K_f)\right)_{\mathrm{cusp}}$$

$$= \underbrace{\bigoplus_{\pi}}_{\text{autom. rep'n of wt } k_\tau\text{, cusp.}} \pi_f^{K_f} \otimes \underbrace{H^1\left(\mathfrak{gl}_2, \mathbb{R}^\times \cdot SO(2); \pi_\infty \otimes \left(\mathrm{Sym}^{k-2}\right)^*\right)}_{\text{2-dim'l.}}$$

$$\rightsquigarrow 0 \to H^0\left(\mathfrak{q}, K_\infty; \pi_\infty \otimes \chi_{k-2}\right) \to H^1\left(\mathfrak{gl}_2, K_\infty; \pi_\infty \otimes \left(\mathrm{Sym}^{k-2}\right)^*\right) \to H^1\left(\mathfrak{q}, K_\infty; \pi_\infty \otimes \chi_{-k}\right) \to 0$$

<u>Langlands observation</u>: $\dim H^{\mathrm{mid}}\left(\mathfrak{g}, K_\infty; \pi_\infty \otimes V(\lambda)\right) = \dim\left(\text{rep of } \hat{G} \text{ of highest wt } \mu\right)$

or rather this is how Langlands discovered the dual group.