READING SEMINAR ON GEOMETRY OF SHIMURA VARIETIES

The goal of this seminar is to understand Helm's geometric construction [He12] of Jacquet–Langlands correspondence in the case of $U(2)$-Shimura varieties. Along the way, we will touch base with topics including PEL type Shimura varieties and its mod p fibers, (partial) Hasse invariants, mod p (Hilbert) modular forms, Katz modular forms, overconvergent (Hilbert) modular forms, Grothendieck–Messing–Serre-Tate theorem on deformation of abelian varieties.

Prerequisite of this seminar includes algebraic geometry, modular forms and modular curves, and basic sense of Langlands correspondence.

Here is the detailed outline of each talk, which typically lasts 1.5-2 hours.

Talk 1 (Liang Xiao)**.** Introduction to modular curves and Siegel modular varieties.

In this talk, we will discuss basics of moduli space of abelian varieties with additional structures: PEL for polarizations, endomorphisms, and level structure. We will start with the case of modular curve, then moves to the Siegel moduli space.

Talk 2. Unitary and Hilbert modular varieties and geometric Hilbert modular forms.

Follow the notes [notes]. Continued with the previous talk, we introduce the definition of unitary Shimura varieties and Hilbert modular varieties. Then we move to explain modular forms in terms of sections of automorphic line bundles. After this, we generalize this to the case of Hilbert modular varieties and possibly general Shimura varieties.

Talk 3. Geometric properties of modular forms.

Follow the notes [notes]. We start with Katz' definition of modular forms, and then use it to define Hecke operators. Also, lightly introduce the notion of Tate curves, and see that evaluating Katz modular forms on Tate curves gives the q-expansion. Then we introduce Kodaira–Spencer isomorphism. At the end, we discuss the relation between de Rham cohomology of local systems and space of modular forms.

Talk 4. Overconvergent modular forms.

Follow notes [notes]. Explain the Hasse invariant and supersingular locus. Discuss the rigid analytic fiber of the ordinary locus and supersingular locus. Define Katz p-adic modular forms, and overconvergent modular forms. Define U_p-operators on these forms and quickly mention Coleman's classicality result. If time permits, discuss the Hilbert case.

Talk 5. Arithmetic compactification of Hilbert modular varieties.

More carefully discuss the degeneration of elliptic curves at the cusp. Then we move to the Hilbert modular varieties. Mostly follow [Ra78]; admits necessary algebraic geometric facts. We may possibly provide some notes [notes] on this.

Talk 6. General theory of Shimura varieties.

Follow the notes [notes]. We will introduce basic theory for Shimura varieties, the theory of Shimura reciprocity law. Classical references include Milne's notes [Milne], Deligne's Bourbaki talk [De71], but they are not so easy to understand.

Talk 7. Conjectures on étale cohomology of Shimura varieties

Introduce automorphic local systems, vector bundles, and canonical P-torsors on a Shimura varieties. Explain the conjectural description of various cohomology of the Shimura varieties (in the nicest situation). Maybe mention phenomenons like non-tempered classes and endoscopy cases, in examples, but will not go into discussing them.

Talk 8. Carayol's construction.

Introduce Carayol's construction of integral models for Shimura curves using unitary Shimura varieties, via "modèles étranges". Essentially follow [Ca86] or [Sa09], it is discuss to consider the case when F is of degree 2 and p splits. No need to go into the application, simply focus on relating quaternionic Shimura curves and unitary Shimura curves.

Talk 9. Shimura varieties of PEL type d'après Kottwitz.

Mostly follow [He12, §2] and some additional content from [notes] which is derived from the classical reference [Ko92]. The purpose of this is to carefully review general theory of Shimura varieties of PEL type and its moduli problems. We will spend some extra time to discuss some subtleties in the theory of Shimura varieties.

Talk 10. Goren–Oort stratification.

Mostly follow [He12, §3]. It is quite well-written. A writing with related content is [HTX17, §3]. This talk will discuss Goren–Oort stratification induced by partial Hasse invariants, and discuss the smoothness of the intersection of the zero loci of these partial Hasse invariants. Spend time to carefully explain the decomposition of Lie algebra and the meaning of relative Frobenius and Verschiebung map. Extremely carefully explain the proof of [He12, Proposition 3.4] using Grothendieck–Messing–Serre–Tate deformation theory of abelian varieties. This is the core technique and will be used over and over later.

Talk 11. Description of sparse Goren–Oort strata

Follow closely [He12, §4]. This is probably the most interesting part of this paper, in which Helm give a *global* description of some Goren–Oort strata, essentially only using the Grothendieck–Messing–Serre–Tate deformation theory of abelian varieties.

Talk 12. Geometry of unitary Shimura varieties with $\Gamma_0(p)$-level structure.

Follow closely [He12, §5 and §6]. It describes certain strata of the unitary Shimura varieties with $\Gamma_0(p)$-level structure, using essentially similar techniques.

Talk 13. Geometric construction of Jacquet–Langlands correspondence.

Follow [He12, §7]. Using Rapoport–Zink weight spectral sequence and the geometric description above to construct Jacquet–Langlands correspondence geometrically. Spend a large portion of the time to discuss Rapoport–Zink weight spectral sequence, following [Sa03]. (It may not be a good idea to develop the theory in full generality, maybe illustrate the RZ spectral sequence in the case of surfaces and threefolds.)

References

[AG05] F. Andreatta and E. Goren, Hilbert modular forms: mod p and p-adic aspects. *Mem. Amer. Math. Soc.* **173** (2005), no. 819, vi+100 pp.

[Ca86] H. Carayol, Sur la mauvaise réduction des courbes de Shimura, *Compositio Math.* **59** (1986), no. 2, 151–230.

[De71] P. Deligne, Travaux de Shimura, Travaux de Shimura. Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, pp. 123–165. Lecture Notes in Math., Vol. 244, Springer, Berlin, 1971.

[Go02] E. Goren, *Lectures on Hilbert modular varieties and modular forms. CRM Monograph Series*, **14**. American Mathematical Society, Providence, RI, 2002. x+270 pp.

[He12] D. Helm, A geometric Jacquet-Langlands correspondence for $U(2)$ Shimura varieties, *Israel. J. Math.* **187** (2012), 37–80.

[HTX17] D. Helm, Y. Tian, and L. Xiao, Tate cycles on some unitary Shimura varieties mod p, *Alg. and Number Theory* **11** (2017), 2213–2288.

[Milne] J. Milne, Introduction to Shimura varieties, notes available on his webpage.

[notes] Written notes for this seminar.

[Ko92] R. Kottwitz, Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.* **5** (1992), no. 2, 373–444.

[Ra78] M. Rapoport, Compactifications de l'Espace de Modules de Hilbert-Blumenthal, *Compositio Math.* **36** (1978), 255–335.

[Sa03] T. Saito, Weight spectral sequences and independence of ℓ, J. of the Inst. of Math. of Jussieu, **2** (2003), 583–634.

[Sa09] T. Saito, Hilbert modular forms and p-adic Hodge theory, *Compos. Math.* **145** (2009), 1081–1113.