2021 Fall Honors Algebra Practice Final Exam.

Comments on the actual exam:

- The actual exam will be in both Chinese and English, as in the midterm.
- The actual exam will contain 10 True/False questions (total 10 points), for the study of those problems, see homeworks.
- Aside from the true/false problems, the distribution of the points is roughly 20% pure group theory, 20% pure ring theory, the rest on field theory. But some field theory questions are secretly questions on groups.
- The layout of the actual exam is 10 points for True/False questions, then 4 standard questions ($10+12+13+15$ points), and then 4 problems (10 points each) with some challenge, and finally two more difficult problems (5 points each). One cannot be awarded more than 100 points out of the 110 points on the exam. Of course, the difficulty of the problems is a rather subjective criterion.

Practice Final exam, minus the True/False questions.

Problem I (10 points) Let G be a group of order 56. Let P_2 and P_7 be 2-Sylow and 7-Sylow subgroups, respectively.

(1) Show that if P_2 and P_7 are both normal then $xy = yx$ for $x \in P_2$ and $y \in P_7$.
(2) Show that either P_2 or P_7 is normal.

Problem II (12 points)

Let R be a PID and let M be a finitely generated R-module. Let F be the field of fraction of R. Show that $\mathrm{Hom}_R(M, F)$ and $M \otimes_R F$ have the same dimension as vector spaces over F.

Problem III (13 points)

In this problem, we will construct a Galois extension with Galois group isomorphic to $G = Z_7 \rtimes Z_3$.

(1) Let σ denote the generator of Z_3 (corresponding to 1 mod 3) and let τ denote the generator of Z_7 (corresponding to 1 mod 7). Give two different maps $\varphi_1, \varphi_2 : Z_3 \to \mathrm{Aut}(Z_7)$, and show that the semi-direct products $Z_7 \rtimes_{\varphi_1} Z_3$ and $Z_7 \rtimes_{\varphi_2} Z_3$ are isomorphic. (We henceforth denote $Z_7 \rtimes Z_3$ for this group.)
(2) Show that the splitting field E of $\sqrt[7]{5}$ over $\mathbb{Q}$ is isomorphic to $Z_7 \rtimes (\mathbb{Z}/7\mathbb{Z})^\times$.
(3) Find a subfield F of E so that $\mathrm{Gal}(E/F) \cong Z_7 \rtimes Z_3$.

Problem VI (15 points)

Let $\alpha = \sqrt{2+\sqrt{2}}$. Write out the splitting field of $\mathbb{Q}(\alpha)$ and draw the diagram of correspondence of Galois groups.

Problem V (10 points) Let $G = \mathrm{SL}_3(\mathbb{F}_p)$, where p is an odd prime. Let ℓ be a prime divisor of p^2+p+1.

(1) Suppose $\ell > 3$. Prove that the ℓ-Sylow subgroups of G are cyclic.

(2) Suppose that $\ell = 3$. Prove that the ℓ-Sylow subgroup of G are *not* cyclic.

Problem VI (10 points) Let F be a field of characteristic 0 and let E be a finite Galois extension of F.

(1) If $0 \neq \alpha \in E$ with $E = F(\alpha)$, show that $F(\alpha^2) \neq E$ if and only if there exists an automorphism $\sigma \in \mathrm{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$.

(2) Prove that there exists an element $\alpha \in E$ with $E = F(\alpha^2)$.

Problem VII (10 points)

(1) Let I, J, K be ideals of a principal ideal domain R. Prove that $I \cap (J+K) = (I \cap J) + (I \cap K)$.

(2) Find ideals I, J, K of $S = \mathbb{C}[x,y]/(x,y)^2$ such that $I \cap (J+K) \neq (I \cap J) + (I \cap K)$. (Hint: You may assume that $(x,y)/(x,y)^2$ is a vector space of dimension 2, thought of as a module over $\mathbb{C} \cong \mathbb{C}[x,y]/(x,y)$.)

Problem VIII (10 points) Let $F \subseteq E$ be finite fields, where $|F| = q < \infty$ and $[E:F] = n$.

(1) Prove that every monic irreducible polynomial in $F[X]$ of degree dividing n is the minimal polynomial over F of some element of E.

(2) Compute the product of all the monic irreducible polynomials in $F[X]$ of degree dividing n.

(3) Suppose $|F| = 2$. Determine the number of monic irreducible polynomials of degree 10 in $F[X]$.

Problem IX (5 points) Let $F \subseteq E$ be fields of characteristic zero and suppose $0 \neq \alpha \in E$ with $E = F(\alpha)$. Assume that some power of α lies in F and let n be the smallest positive integer such that $\alpha^n \in F$.

(1) If $\alpha^m \in F$ with $m > 0$, show that m is a multiple of n.

(2) If every root of unity in E lies in F, show that $[E:F] = n$.

Problem X (5 points)

Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial with splitting field E over F. Choose $\alpha \in E$ with $f(\alpha) = 0$. Furthermore, for some fixed integer $n \geq 1$, let $g(x)$ be an irreducible polynomial in $F[x]$ with $g(\alpha^n) = 0$.

(1) Show that $\deg(g)$ divides $\deg(f)$ and that $\deg(f)/\deg(g) \leq n$.

(2) If $\deg(f)/\deg(g) = n$ and if the characteristic of F does not divide n, prove that E contains a primitive nth root of unity.