2023 Fall Honors Algebra Homework 1 (due on September 28)

For submitted part, please complete all True/False questions, choose 10 out of the standard questions, and 5 out of the more challenging questions.

[A] = Artin, [DF] = Dummit and Foote, [DN] = Ding and Nie (Chinese), [H] = Hungerford, [Ja] = Jacobson, [XZ] = Xu and Zhao

1.1. **True/False questions.** (Only write T or F when submitting the solutions. Put the solutions five in a row for the convenience of the graders)

(1) $(\mathbb{Q}\backslash\{0\}, *)$ with $a * b = \frac{ab}{2}$ defines a commutative group structure on $\mathbb{Q}\backslash\{0\}$. (What does it look like?)
(2) If $\phi : G \to G$ is a homomorphism from a group G to itself, if ϕ is surjective, then ϕ is an isomorphism.
(3) In a group G, for any $f, g, h \in G$, the equation $f \cdot x \cdot g = h$ has a solution x in G.
(4) In a cyclic group, every element is a generator.
(5) If H, K are subgroups of a group G, then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup.
(6) The empty set can be viewed as a group.
(7) For any two groups G and G', there exists a homomorphism $\phi : G \to G'$. (Think carefully.)
(8) A homomorphism may have empty kernel.
(9) It is not possible to have a nontrivial homomorphism from a finite group to an infinite group.
(10) Every quotient group of a nonabelian group is nonabelian.
(11) Let G be a group, then the quotient G/G is the empty set.
(12) $\mathbb{R}/n\mathbb{R}$ is a cyclic group of order n, where $n\mathbb{R} = \{nr \mid r \in \mathbb{R}\}$ and $\mathbb{R}$ is under addition.
(13) Let G be a group and N a normal subgroup. Suppose that K and K' are conjugate subgroups of G, then KN and $K'N$ are conjugate subgroups of G.
(14) Every finite group of prime order is solvable.
(15) Two groups with the same set of Jordan–Hölder factors are isomorphic.

1.2. **Warm-up problems.** (Do not submit; exercise on your own.)

Problem 1.2.1. [DF, page 22, problem 9]
Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.

(a) Prove that G is a group under addition.
(b) Prove that the nonzero elements of G are a group under multiplication.

Problem 1.2.2. [DF, page 22, problem 15]
Prove that, for elements $a_1, \dots, a_n$ in a group G, $(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$.

Problem 1.2.3. [DF, page 22, problems 18 and 25]
Part I: Let x and y be elements of a group G. Prove that the following are equivalent:

(1) $xy = yx$ (i.e. x and y commutes)
(2) $yxy^{-1} = x$ (i.e. the conjugate of x by y is still x)
(3) $xyx^{-1}y^{-1} = 1$ (i.e. the *commutator* of x and y is 1)

Part II: Prove that if $x^2 = 1$ for all $x \in G$, then G is abelian.

Problem 1.2.4. [DF, page 34, problems 18 and 19]
Find all numbers n such that S_5 contains an element of order n. Do the same for S_7.

Problem 1.2.5. [DF, page 40, problem 18]
Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

Problem 1.2.6. [DF, page 49, problem 14]
Show that $\{x \in D_{2n} \mid x^2 = 1\}$ is not a subgroup D_{2n} (here $n \geq 3$).

Problem 1.2.7. [DN, page 54, problem 8] Prove that a group G is a commutative group if and only if the map $x \mapsto x^{-1}$ is an isomorphism of groups.

Problem 1.2.8. [H, page 37, problem 6]
If G is a cyclic group of order n and $k|n$, then G has exactly one subgroup of order k.

Problem 1.2.9. [DF, page 87, problem 17]
Let G be the dihedral group of order 16

$$G = \langle r, s \mid r^8 = s^2 = 1, rs = sr^{-1} \rangle$$

and let $\overline{G} := G/\langle r^4 \rangle$ be the quotient of G by the subgroup generated by r^4.

(1) Show that $\langle r^4 \rangle$ is normal; and show that $\#\overline{G} = 8$.
(2) Exhibit each element of $\overline{G}$ in the form $\bar{s}^a \bar{r}^b$, for some integers a and b.
(3) Give an explicit isomorphism $\overline{G} \to D_8$.

Problem 1.2.10. Let F be a field (or just simply $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$). Consider matrix groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in F,\, a \neq 0, d \neq 0 \right\} \supset N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in F \right\}$$

(1) Show that N is a normal subgroup of G.
(2) Express each element of the quotient G/N by elements in G.

Problem 1.2.11. [DN, page 55, problem 16]
In the group $\mathrm{SL}_2(\mathbb{Q})$, show that the element $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has order 4 and the element $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ has order 3, yet ab has infinite order.
(So even though a and b have finite order, ab can have infinite order.)

Problem 1.2.12. [DF, page 95, problem 8]
Prove that if H and K are finite subgroups of G whose orders are relatively prime then $|H \cap K| = 1$.

Problem 1.2.13. [DF, page 96, problem 11]
Let $H \leqslant K \leqslant G$. (**Do not assume that G is finite.**) Prove that

$$|G : H| = |G : K| \cdot |K : H|.$$

Problem 1.2.14. [DF, page 101, problem 4]
Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that

$$(C \times D) \trianglelefteq (A \times B) \qquad \text{and} \qquad (A \times B)/(C \times D) \simeq (A/C) \times (B/D).$$

Problem 1.2.15. [XZ, page 24, 15]
Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{2\pi i/n} \end{smallmatrix}\right)$. Prove that the set $\{I, B, B^2, \dots, B^{n-1}, A, AB, \dots, AB^{n-1}\}$ form a group under matrix multiplication. Moreover, this group is isomorphic to D_{2n}.

1.3. **Standard problems.** (Choose 10 problems to submit.)

Problem 1.3.1. [DF, page 22, problem 22] If x and g are elements of a group G, prove that $|x| = |gxg^{-1}|$. Deduce that $|ab| = |ba|$ for all $a, b \in G$.

Problem 1.3.2. Let $n, m \in \mathbb{N}_{>2}$. Determine all homomorphisms $\phi : \mathbf{Z}_n \to D_{2m}$.

Problem 1.3.3. Prove that S_n is generated by (12) and $(12 \cdots n)$. (One can probably prove this directly, but one can also prove it as listed in the class, by first showing S_n is generated by all (ij); and then showing S_n is generated by $(i, i+1)$; and finally this statement.)

Problem 1.3.4. [DF, page 33, problem 13 and partially problem 17] Show that an element has order 2 in S_n if and only if its cycle decomposition is a product of commuting 2-cycles. In S_6, how many elements have order 2?

Problem 1.3.5. [DF, page 40, problem 7]
Prove that D_8 and Q_8 are not isomorphic. (Find the definition in Dummit–Foote.)

Problem 1.3.6. Prove that every subgroup of a cyclic group is cyclic, and every quotient group of a cyclic group is cyclic.

Problem 1.3.7. [A, page 68, 2.6 and page 70, 6.9] Let G be a group. Define an *opposite group* G^{op} with law of composition $a \star b$ as follows: the underlying set is the same as G, but the law of composition is $a \star b = ba$.

- Prove that G^{op} is a group.
- Prove that G and G^{op} are isomorphic.

Problem 1.3.8. Let $\varphi : G \to H$ be a group homomorphism, and let K be a subgroup of H.

(1) Show that $\varphi^{-1}(K)$ is a subgroup of G.
(2) When E is a normal subgroup of H, show that $\varphi^{-1}(E)$ is normal in G.
(3) When G is finite, show that $|\varphi^{-1}(K)| = |\ker(\varphi)| \cdot |\varphi(G) \cap K|$. (Note: one needs to first show the finiteness of each term.)

Problem 1.3.9. [DF, page 87, problem 21]
Consider the group $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ given in terms of the following generators and relations
$$G = \langle x, y \mid x^4 = y^4 = 1, xy = yx \rangle.$$
Let $\overline{G} = G/\langle x^2y^2 \rangle$ (note that every subgroup of the abelian group G is normal).

(1) Show that the order of $\overline{G}$ is 8.
(2) Exhibit each element of $\overline{G}$ in the form of $\bar{x}^a \bar{y}^b$ for some integers a and b.
(3) Find the order of each of the elements of $\overline{G}$ exhibited in (2).
(4) Give an explicit isomorphism $\overline{G} \simeq \mathbf{Z}_4 \times \mathbf{Z}_2$.

Problem 1.3.10. [DF, page 88, problem 27]
Let N be a *finite* subgroup of a group G. Show that for an element $g \in G$, $gNg^{-1} \subseteq N$ if and only if $gNg^{-1} = N$. Can you give an example where this fails for an infinite N? (Hint: can consider a case similar to Problem 1.2.10.)

Problem 1.3.11. [DF, page 88, problem 34]
Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ be the usual presentation of dihedral group of order $2n$, and let k be a positive integer dividing n.

(1) Prove that $\langle r^k \rangle$ is a normal subgroup of D_{2n}.
(2) Prove that $D_{2n}/\langle r^k \rangle \cong D_{2k}$.

Problem 1.3.12. [DN, page 55, problem 12]
Any subgroup H of G of index 2 is normal.

Problem 1.3.13. [DN, page 97, problem 15]
Let H_1, H_2 be subgroups of a group G, and let N be a normal subgroup of G contained in $H_1 \cap H_2$. Show that

$$\big(H_1/N\big) \cap \big(H_2/N\big) \cong \big(H_1 \cap H_2\big)/N.$$

Here the intersection takes place in G/N.

Problem 1.3.14. [DN, page 98, problem 27]
Let G be a finite group. Let $H, K < G$ be subgroups and let $a \in G$. The set

$$HaK := \{hak \mid h \in H,\, k \in K\}$$

is called a double cosets for H and K. Prove that

$$|HaK| = |H| \cdot [K : a^{-1}Ha \cap K].$$

Problem 1.3.15. [H, page 41, problem 10]
Let H, K, N be subgroups of a group G such that

$$H < K, \quad H \cap N = K \cap N, \quad \text{and} \quad HN = KN.$$

Show that $H = K$.

Problem 1.3.16. [DF, page 96, problem 10]
Suppose that H and K are subgroups of finite index in the (possibly infinite) group G with $[G : H] = m$ and $[G : K] = n$. Prove that

$$\operatorname{lcm}(m, n) \leq [G : H \cap K] \leq mn.$$

Deduce that if m and n are relatively prime, then

$$[G : H \cap K] = [G : H] \cdot [G : K].$$

Problem 1.3.17. [XZ, page 58, problem 19]
Suppose that H and K are both normal subgroups of G and that G/H and G/K are both solvable. Prove that $G/(H \cap K)$ is solvable.

Problem 1.3.18. (online sources)
(1) Let G be a finite abelian group with elements $a_1, a_2, \ldots, a_n$. (But we use multiplicative convention in this problem.) Prove that $a_1a_2 \cdots a_n$ is an element whose square is the identity.
(2) If the G in part (1) has no elements of order 2 or more than one elements of order 2. Prove that $a_1a_2 \cdots a_n = e$.
(3) If G has exactly one element y of order 2, prove that $a_1a_2 \cdots a_n = y$.
(4) (Wilson's theorem) If p is a prime number, show that $(p-1)! \equiv -1 \pmod{p}$.

Problem 1.3.19. Let G be a group and let A be a subgroup and N a normal subgroup. Prove that AN is solvable if and only if both A and N are solvable.

1.4. **More challenging questions.** (Choose 5 problems to submit.)

Problem 1.4.1 (Subgroups of dihedral groups)**.** List all subgroups of the dihedral group D_{2n} (note: one has to separate the discussion by whether n is even or odd). Determine which subgroup is normal.

Problem 1.4.2. [Jacobson, page 53, problem 6]
Let H be a subgroup of a finite group G. Show that there exist elements $z_1, \dots, z_n$ simultaneously representing left and right cosets of H in G, that is

$$G/H = \{c_1H, \dots, c_nH\} \quad \text{and at the same time} \quad H\backslash G = \{Hc_1, \dots, Hc_n\}.$$

(There is a hint in that book. I leave it to you to decide whether to look at it.)

Problem 1.4.3. [DF, problems 10 and 11]
Let G be the group of rigid motions in $\mathbb{R}^3$ of a cube (or an octahedron). Show that $|G| = 24$. Give an explicit isomorphism $G \simeq S_4$.

Problem 1.4.4. [H, page 29 problem 8, page 30 problem 10, page 37, problem 7]

(1) Give a one-line proof of that $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}$ (the additive subgroup).
(2) Let p be a prime and let $Z(p^\infty)$ be the following subset of $\mathbb{Q}/\mathbb{Z}$:
$$Z(p^\infty) = \big\{\overline{a/b} \in \mathbb{Q}/\mathbb{Z} \;\big|\; a, b \in \mathbb{Z} \text{ and } b = p^i \text{ for some } i \geq 0\big\}.$$
Show that $Z(p^\infty)$ is an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$.
(3) Show that every element of $Z(p^\infty)$ has finite order p^n for some $n \geq 0$.
(4) Show that the subgroup $H_n := \{z \in Z(p^\infty) \mid p^n \cdot z = 0\}$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$.
(5) Deduce that $Z(p^\infty) = \bigcup_n H_n$.
(6) Prove that $Z(p^\infty)$ is divisible, that is, for each element $x \in Z(p^\infty)$ and for each $n \in \mathbb{N}$, there exists $y \in Z(p^\infty)$ such that $x = ny$.

Problem 1.4.5. Prove the fourth isomorphism theorem (on your own): Let G be a group and let N be a normal subgroup of G; write $\pi : G \to \overline{G} = G/N$ denote the projection. Show that there is a bijection

$$\begin{array}{rcl} \big\{\text{subgroups } A \text{ of } G \text{ containing } N\big\} & \longleftrightarrow & \big\{\text{subgroups of } \overline{G}\big\} \\ A & \longmapsto & \overline{A} := A/N \\ \pi^{-1}(C) & \longleftarrow\!\shortmid & C \end{array}$$

In addition, prove that, for all $A, B \leqslant G$ with $N \leqslant A$ and $N \leqslant B$

(1) $A \leqslant B$ if and only if $\overline{A} \leqslant \overline{B}$,
(2) $A \leqslant B$, then $[B : A] = [\overline{B} : \overline{A}]$,
(3) $\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle$,
(4) $\overline{A \cap B} = \overline{A} \cap \overline{B}$,
(5) $A \trianglelefteq G$ if and only if $\overline{A} \trianglelefteq \overline{G}$.

Problem 1.4.6. [H, page 51, problem 7]
Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4 contained in A_4. Give explicit isomorphisms $S_4/N \cong S_3$ and $A_4/N \cong \mathbb{Z}/3\mathbb{Z}$. (Read the definition of A_n on your own.)

Problem 1.4.7. [A, page 77, problems 7–9]

The set of $n \times n$ matrices can be identified with the space $\mathbb{R}^{n\times n}$. Let G be a subgroup of $\mathrm{GL}_n(\mathbb{R})$. Prove the following:

(1) If $A, B, C, D \in G$ and if there are paths in G from A to B and from C to D, then there is a path from AC to BD. (Here a path from A to B means a continuous function $f : [0,1] \to G$ such that $f(0) = A$ and $f(1) = B$.)

(2) The set of matrices that admits a path to the identify I forms a normal subgroup of G (called the connected component of G), often denoted by G°.

(3) Show that $\mathrm{SL}_n(\mathbb{R})$ is generated by elementary matrices of the form $I + aE_{ij}$ for $a \in \mathbb{R}$, $i \neq j$ and E_{ij} the $n \times n$-matrix with all zero entries except at (i,j)-entry, where it is 1.

(4) Use above to deduce that $\mathrm{SL}_n(\mathbb{R})$ is path-connected (that is, all elements are linked to I_n by paths).

(5) Show that the subgroup of connected component $\mathrm{GL}_n(\mathbb{R})^\circ$ in $\mathrm{GL}_n(\mathbb{R})$ has index 2.

Problem 1.4.8 (Yau contest 2011)**.** Consider the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) \mid ad - bc = 1 \right\}.$$

Show that it is generated by two matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Problem 1.4.9 (Yau contest 2014)**.** Let S and T be nonabelian finite simple groups, and write $G = S \times T$.

(1) Show that the total number of normal subgroups of G is four.

(2) If S and T are isomorphic, show that G has a maximal proper subgroup not containing either direct factor (meaning $\{1\} \times T$ or $S \times \{1\}$). (A *maximal proper subgroup* of G is a subgroup H of G such that $H \neq G$ and the only subgroup of G strictly containing H is G itself. The existence of maximal proper subgroups is clear in this context because G is finite; you need to show that, there is one such maximal proper subgroup that does not contain S nor T.)

(3) If G has a maximal proper subgroup that contains neither of the direct factors of G, show that S and T are isomorphic.

Problem 1.4.10 (Yau contest 2017)**.** Let G be a group and let $g \in G$ be an element of finite order n. Suppose that n is the product of two positive integers r and s which are coprime to each other.

(1) Show that there is a pair (g_1, g_2) of elements of G such that $g_1^r = g_2^s = 1$ and $g_1 g_2 = g_2 g_1 = g$.

(2) Show that such a pair is unique.