2023 Fall Honors Algebra Additional Exercises (just for fun)

[A] = Artin, [DF] = Dummit and Foote, [DN] = Ding and Nie (Chinese), [H] = Hungerford.

All rings in this section are **commutative** with 1.

8.1. **True/False questions.** (Only write T or F when submitting the solutions.)

(1) If I and J are ideals in a ring R such that $\sqrt{I}$ and $\sqrt{J}$ are comaximal, then I and J are comaximal.
(2) A prime ideal of a ring cannot contain a unit.
(3) Let k be an algebraically closed field. If $Z_1, Z_2 \subseteq k^n$ are algebraic subsets, then $Z_1 \cup Z_2$ is an algebraic subset.
(4) If $R \subseteq S$ is an integral extension of integral domains, then R is a field if and only if S is a field.
(5) The extension $\mathbb{Z}[x^3+2x^2+2]$ is an integral extension of $\mathbb{Z}[x]$.

Problem 8.1.1. For ideals $I, J \subseteq R$, we have

(1) $I \subseteq \sqrt{I}$;
(2) $\sqrt{\sqrt{I}} = \sqrt{I}$;
(3) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$;
(4) $\sqrt{I} = (1) \Leftrightarrow I = (1)$;
(5) $\sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}}$;
(6) if $\mathfrak{p}$ is a prime ideal, then $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n > 0$.

Problem 8.1.2. Given an example of a ring R and ideals I and J such that $\sqrt{I+J} \neq \sqrt{I} + \sqrt{J}$. See Problem 8.1.1(5).

Problem 8.1.3. For $n \in \mathbb{Z}$ that is not zero and is square-free. Compute the integral closure $\mathcal{O}$ of $\mathbb{Z}$ inside $\mathbb{Q}(\sqrt{n})$. Explicitly,

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\sqrt{n}] & n \equiv 2, 3 \bmod 4 \\ \mathbb{Z}[\frac{\sqrt{n}+1}{2}] & n \equiv 1 \bmod 4. \end{cases}$$

Problem 8.1.4. A ring with exactly one maximal ideal is called a *local ring*.

(1) Let A be a ring and $\mathfrak{m} \neq (1)$ an ideal of A such that every $x \in A - \mathfrak{m}$ is a unit in A. THen A is a local ring and $\mathfrak{m}$ its maximal ideal.

(2) Let A be a ring and $\mathfrak{m}$ a maximal ideal of A, such that every element of $1+\mathfrak{m}$ is a unit in A. Then A is a local ring.

Problem 8.1.5. Let k be a field. Show that $k[t]$ is integrally closed in $k(t)$.

Problem 8.1.6. Let $R \subseteq S$ be an integral ring extension. Show that $R[x] \subseteq S[x]$ is an integral extension.