# Recognizing direct products, group actions, semi-direct products

## Recognizing direct products

**Theorem** Suppose $G$ is a group with subgroups $H$ and $K$ such that

(1) $H$ and $K$ are normal in $G$, and

(2) $H \cap K = \{e\}$

Then $HK = H \times K$

**Proof:** $H, K \trianglelefteq G \Rightarrow HK$ is a normal subgroup of $G$

Consider the map $\varphi : H \times K \longrightarrow HK$

$$(h, k) \longmapsto hk$$

$\varphi$ is a homomorphism : $\varphi\big(\underbrace{(h_1,k_1)(h_2,k_2)}_{= (h_1h_2,\, k_1k_2)}\big) \overset{?}{=} \varphi(h_1,k_1)\cdot\varphi(h_2,k_2)$

$$\Leftrightarrow \quad h_1h_2k_1k_2 \overset{?}{=} h_1k_1h_2k_2$$

$$\Leftrightarrow \quad h_2k_1 \overset{?}{=} k_1h_2$$

$$\Leftrightarrow \quad h_2k_1h_2^{-1}k_1^{-1} \overset{?}{=} e$$

But $h_2\underbrace{k_1h_2^{-1}k_1^{-1}}_{\text{in } H} \in H$ , $\underbrace{h_2k_1h_2^{-1}}_{\text{in } K}k_1^{-1} \in K \Longrightarrow h_2k_1h_2^{-1}k_1^{-1} \in H \cap K = \{e\}$

$\varphi$ is clearly surjective

$\ker\varphi = \{(k,h) \in K \times H \mid kh = e\}$ $\Rightarrow \ker\varphi = \{e\}.$

$$\Downarrow$$

$$k = h^{-1} \in H \cap K = \{e\}$$

So $\varphi$ is an isomorphism.

## Group action:

(Motivating example: $S_n$ moves elements in $\{1,\dots,n\}$ around.)

**Definition.** Let $G$ be a group and $X$ a set. A (left) <u>$G$-action on $X$</u> (G在X上左作用)

is a map
$$G \times X \longrightarrow X$$
$$(g,x) \longmapsto g\cdot x$$

such that (1) $\forall x \in X$, $e\cdot x = x$

(2) For $g,h\in G$ and $x\in X$, $g\cdot(h\cdot x) = gh\cdot x$

We sometimes write $G \circlearrowright X$

**Remark** The conditions imply that $\forall g\in G$, $\begin{array}{l} X \longrightarrow X \\ x \longmapsto g\cdot x \end{array}$ is a bijection

(b/c $g^{-1}$ is its inverse.)

**Examples** ① $S_n \circlearrowright \{1,\dots,n\} = X$

b/c $\sigma(\tau(i)) = (\sigma\cdot\tau)(i)$

It also induces an action $S_n \circlearrowright \{A\subseteq X \mid \# A = 2\}$

② $G$-action on itself:

* left translation: $g\in G \rightsquigarrow l_g : G \longrightarrow G$
$$l_g(x) := gx$$

* right translation: $g\in G \rightsquigarrow r_g : G\longrightarrow G$
$$r_g(x) := x\cdot g^{-1}$$

Why $g^{-1}$?? b/c we need $r_g\circ r_h(x) = r_{gh}(x)$ $\forall x\in G$

$$r_g\circ r_h(x) = r_g(xh^{-1}) = xh^{-1}g^{-1} = x(gh)^{-1} = r_{gh}(x) \checkmark$$

If we defined $r'_g(h) = hg$, then $r'_{gh}(x) = xgh = r'_h(xg) = r'_h r'_g(x) \underset{\text{in general}}{\neq} r'_g r'_h(x)$

* conjugation action: $g\in G \rightsquigarrow Ad_g : G\longrightarrow G$

$$Ad_g(x) = gxg^{-1}$$

This is a "better" action: $Ad_g(xy) = gxyg^{-1} = gxg^{-1}\cdot gyg^{-1} = Ad_g(x)\cdot Ad_g(y)$

i.e. $Ad_g$ preserves the group structure $\Rightarrow$ it's a homomorphism.

· <u>Definition</u> A <u>right action</u> (右作用) is a map
$$X \times G \longrightarrow X$$
$$(x, g) \longmapsto x\cdot g$$

s.t. $xe = x$ and $(x\cdot g)h = x\cdot gh$

E.g. right multiplication by $g$ is a right action

$$r'_g : G \longrightarrow G$$
$$r'_g(h) = hg$$

<u>Proposition</u> Let $G$ be a group acting on a set $X$. Then we have a natural homomorphism
$$\Phi: G \longrightarrow S_X := \text{permutation group of } X$$
$$g \longmapsto (\phi_g : x \longmapsto gx)$$

In fact, giving a $G$-action on the set $X$ is equivalent to giving such a homomorphism $\Phi$

Proof: Need to check $\phi_g \circ \phi_h = \phi_{gh}$

$$\forall x \in X,\ \phi_g \circ \phi_h(x) = \phi_g(hx) = g(hx) = (gh)(x) = \phi_{gh}(x). \quad \square$$

<u>Definition</u> If this homomorphism $\Phi$ is injective, we say that the action is <u>faithful</u> (忠实的)

(i.e. $\ker\Phi = \{e\}$, meaning no nontrivial element of $G$ fixes all elements of $X$)

If this homomorphism is trivial, i.e. $g \mapsto id$, we say that the action is <u>trivial</u> (平凡的)

<u>Cayley's Theorem</u> Every group is isomorphic to a <u>subgroup</u> of some symmetry group.

If $\# G = n$, then $G$ is isomorphic to a subgroup of $S_n$.

(Proof: the left translation action gives $G \hookrightarrow S_G$)

<u>Remark</u>. ① This theorem has historical meaning b/c groups were first defined as subgroups of $S_n$

Cayley's theorem says our abstract definition agrees with the old definition.

② Given an abstract group, if we want to understand $G$ at the element level,

it is better to let $G$ act on some set $X$, so that we represent $G$ as a subgroup of $S_X$

<u>Definition</u> An <u>automorphism</u> of a group $G$ is an isomorphism $\sigma : G \xrightarrow{\sim} G$

$\mathrm{Aut}(G) := \{\text{automorphisms of } G\}$ form a group

* identity is $\mathrm{id} : G \to G$
* group operation = composition

It is a subgroup of $S_G$ = permutation group of elements in $G$

<u>Example</u>: If we consider the conjugation action $G \overset{\mathrm{Ad}}{\curvearrowright} G$,

it induces $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G) \leq S_G$

$$g \longmapsto (\mathrm{Ad}_g : x \mapsto gxg^{-1})$$

(Have shown, $\forall g \in G$, $\mathrm{Ad}_g : G \to G$ is a homomorphism, hence an isom with inverse $\mathrm{Ad}_{g^{-1}}$)

Ad is a group homomorphism:

$$\mathrm{Ad}(gh) \overset{?}{=} \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$$

$$\Longleftrightarrow \forall x \in G, \quad \underbrace{\underset{\substack{\| \\ ghx(gh)^{-1}}}{\mathrm{Ad}_{gh}(x)} \overset{?}{=} \underset{\substack{\| \\ \mathrm{Ad}_g(hxh^{-1}) = ghxh^{-1}g^{-1}}}{\mathrm{Ad}_g(\mathrm{Ad}_h(x))}}_{=}$$

<u>Remark</u>: A slight generalization of the above scenario is the following:

$X$ itself is a group, $G$ acts on $X$ preserving group structure i.e. $\forall g \in G$, $\phi_g : X \to X$ is a homomorphism/isomorphism

equivalently, the corresponding map $\Phi$:

## Semi-direct product.

Prototypical example $B = \begin{pmatrix} \mathbb{R}^\times & \mathbb{R} \\ 0 & 1 \end{pmatrix} \geq \begin{pmatrix} \mathbb{R}^\times & \\ & 1 \end{pmatrix} = T$

$$\cup\!\!\!\mid$$

$$N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}.$$

$N \trianglelefteq B$ is a normal subgroup. $T \leq B$ is not normal, $B \cap T = \{1\}$

$B = TN \rightsquigarrow$ Under conjugation, $T$ acts on $N$ $\rightsquigarrow B = T \ltimes N$

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & tn \\ 0 & 1 \end{pmatrix}$$

(Reverse eigenineer: construct $B$ out of $T, N$ & the $T$-action on $N$.)

**Definition.** Let $N$ and $H$ be groups and $\varphi : H \longrightarrow \mathrm{Aut}(N)$ a homomorphism

For $h \in H$, write $\varphi_h = \varphi(h) : N \to N$ for the corresponding automorphism

Define the semidirect product (半直积) $N \rtimes H := N \rtimes_\varphi H$ to be

$$\{(n,h) \mid n \in N, h \in H\}$$

$$(n_1, h_1)(n_2, h_2) = (n_1 \cdot \varphi_{h_1}(n_2), h_1 h_2)$$

Check: $((n_1,h_1)(n_2,h_2))(n_3,h_3) \overset{?}{=} (n_1,h_1)((n_2,h_2)(n_3,h_3))$

$= (n_1 \varphi_{h_1}(n_2), h_1 h_2)(n_3, h_3)$ $\qquad = (n_1, h_1)(n_2 \varphi_{h_2}(n_3), h_2 h_3)$

$= (n_1 \varphi_{h_1}(n_2) \varphi_{h_1 h_2}(n_3), h_1 h_2 h_3)$ $\qquad = (n_1 \cdot \varphi_{h_1}(n_2 \varphi_{h_2}(n_3)), h_1 h_2 h_3)$

$= (n_1 \varphi_{h_1}(n_2) \varphi_{h_1}(\varphi_{h_2}(n_3)), h_1 h_2 h_3) = (n_1 \varphi_{h_1}(n_2) \varphi_{h_1}(\varphi_{h_2}(n_3)), h_1 h_2 h_3)$

The sets $N = \{(n,1) \mid n \in N\} \subseteq N \rtimes H$, $H = \{(1,h) \mid h \in H\} \subseteq H \rtimes N$ are subgroups

$\pi : N \rtimes H \longrightarrow H$ is a homomorphism and $N = \ker(\pi)$

$(n,h) \longmapsto h$ $\qquad \Rightarrow N$ is a normal subgroup

Two ways to memorize the notation $N \rtimes H$ : ① $N$ is a normal subgroup

② Group action $G \curvearrowright X$ , $N \rtimes H$ "$N$ is acted on by $H$"

<u>Proposition</u> (Recognizing semidirect product)

Let $G$ be a group, $N \trianglelefteq G$, $H \leq G$, s.t. $H \cap N = \{e\}$

Then $NH$ is a subgroup of $G$ isomorphic to $N \rtimes H$.

Proof: Have shown $NH \leq G$. As $N \trianglelefteq G$, we have $\forall h \in H$, $Ad_h : N \longrightarrow N$, $n \longmapsto hnh^{-1}$

this gives $Ad : H \longrightarrow Aut(N)$

Then $N \rtimes_{Ad} H \xrightarrow{\sim} NH$ as a semidirect product

$(n,h) \longmapsto nh$

<u>Proposition</u> Let $N$ and $H$ be groups and let $\varphi : H \longrightarrow Aut(N)$ be a homomorphism.

TFAE = The Following Are Equivalent:

(1) The identity map between $N \rtimes H$ and $N \times H$ is a group homomorphism (hence an isom.)

(2) $\varphi$ is the trivial homomorphism from $H \to Aut(N)$

(3) $H \trianglelefteq N \rtimes H$.

<u>Example</u>: • $\mathbb{Z}_n^\times = \{ a \bmod n \mid \gcd(a,n) = 1 \} \curvearrowright \mathbb{Z}_n$ by multiplication

$\rightsquigarrow$ semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_n^\times$

$\rightsquigarrow$ can be visualized as $\begin{pmatrix} \mathbb{Z}_n^\times & \mathbb{Z}_n \\ 0 & 1 \end{pmatrix}$

• Take $\{\pm 1\} \subseteq \mathbb{Z}_n^\times \curvearrowright \mathbb{Z}_n$

$\mathbb{Z}_n \rtimes \{\pm 1\} \simeq D_{2n}$

• Group of order $pq$, for $p \mid q-1$, $p, q$ prime

<u>Fact</u>: $\mathbb{Z}_q^\times$ is a cyclic group of order $q-1$

So it contains a unique subgroup of order $p$.

$\rightsquigarrow \mathbb{Z}_q \rtimes \mathbb{Z}_p$ a group of order $pq$

E.g. $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ comes from $\mathbb{Z}_3 \longrightarrow \mathbb{Z}_7^\times$ ← call them $\varphi_2, \varphi_4$

$$b \longmapsto 2^b \quad \text{or} \quad 4^b$$

$$(a_1, b_1)(a_2, b_2) = (a_1 + 2^{b_1} \cdot a_2, b_1 + b_2) \quad \text{or} \quad (a_1 + 4^{b_1} \cdot a_2, b_1 + b_2)$$

Fact: For two different nontrivial homomorphisms

$$\varphi_1, \varphi_2 : \mathbb{Z}_p \longrightarrow \mathbb{Z}_q^\times,$$

the semidirect products $\mathbb{Z}_q \rtimes_{\varphi_i} \mathbb{Z}_p$ are isomorphic

Conclusion All groups of order $pq$ are either isom to $\mathbb{Z}_{pq}$ or to $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.

E.g. $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_3 \xrightarrow{\psi} \mathbb{Z}_7 \rtimes_{\varphi_4} \mathbb{Z}_3$

$$(a, b) \longmapsto (a, 2b)$$

$$\psi((a,b)(c,d)) \stackrel{?}{=} \psi(a,b)\psi(c,d)$$

$$\psi(a + c \cdot 2^b, b+d) \qquad (a, 2b) \cdot (c, 2d)$$

$$(a + c \cdot 2^b, 2b + 2d) \qquad (a + \underset{\substack{\| \\ 16^b = 2^b}}{\underline{4^{2b}}} \cdot c, 2b + 2d)$$

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