2021 秋: 代数学一 (实验班) 期中考试

姓名: ____________ **院系:** ____________ **学号:** ____________ **分数:**

时间：110 分钟 满分：100 分

所有的环都有乘法单位元, 且与其加法单位元不相等; 所有环同态把 1 映到 1.

All rings contains 1_R and $1_R \neq 0_R$; all ring homomorphism takes 1 to 1.

判断题 在下表中填写 T 或 F (10 分)

1	2	3	4	5	6	7	8	9	10

1. 如果 H 是群 G 的正规子群, K 是 H 的正规子群, 那么 K 是 G 的正规子群.

If H is a normal subgroup of G and K is a normal subgroup of H, then K is a normal subgroup of G.

2. 对 $i = 1, 2$, 设 H_i 是 G_i 的正规子群满足 $H_1 \cong H_2$ 和 $G_1 \cong G_2$, 则 $G_1/H_1 \cong G_2/H_2$.

For $i = 1, 2$, let H_i be a normal subgroup of G_i satisfying $H_1 \cong H_2$ and $G_1 \cong G_2$, then $G_1/H_1 \cong G_2/H_2$.

3. 任一非平凡的循环群的非平凡子群一定是循环群.

All nontrivial subgroups of a nontrivial cyclic group is cyclic.

4. 如果 N 是群 G 的正规子群, 则 G 是 N 和 G/N 的半直积.

If N is a normal subgroup of G, then G is a semi-direct product of N with G/N.

5. 若 P 是群 G 的一个西罗 p-子群, 则 P 在 G 中的正规化子是 G 的正规子群.

If P is a Sylow p-subgroup of G, then the normalizer of P in G is normal in G.

6. 两个有限交换群的半直积是可解群.

A semi-direct product of two finite abelian groups is solvable.

7. 群同态 $\varphi : Z_{12} \to Z_{35}$ 必然是平凡的.

A homomorphism $\varphi : Z_{12} \to Z_{35}$ of groups must be the trivial homomorphism.

8. 整环的子环一定是整环.

A subring of an integral domain is an integral domain.

9. 两个整环的直积还是整环.

The direct product of two integral domains is again an integral domain.

10. 若 R 是一个主理想整环, 则 $R[x]$ 是一个主理想整环.

If R is a PID, then $R[x]$ is a PID.

解答题一 (10 分) 证明: 阶为 132 的群不是单群.
Prove that no simple group has order 132.

解答题二 (10 分) 设 $\varphi : R \to S$ 为两个交换环之间的同态.

(1) 证明: 若 P 是一个 S 的素理想, 则 $\varphi^{-1}(P)$ 是 R 的一个素理想.

(2) 证明: 若 M 是 S 的一个极大理想且 φ 是满射, 则 $\varphi^{-1}(M)$ 是 R 的一个极大理想.

(3) 给出一个例子说明 (2) 在不假设 φ 满射时不成立.

Let $\varphi : R \to S$ be a homomorphism of commutative rings.

(1) Prove that if P is a prime ideal of S, then $\varphi^{-1}(P)$ is a prime ideal of R.

(2) Prove that if M is a maximal ideal of S and φ is surjective, then $\varphi^{-1}(M)$ is a maximal ideal of R.

(3) Give an example to show that (2) does not hold without assuming φ to be surjective.

解答题三 (10 分) 记 R 为一整环, F 为其分式域. 对 F 中任一元素 q, 定义 $I_q := \{r \in R \,|\, rq \in R\}$.

(1) 证明: I_q 是环 R 的一个理想.

(2) 现设 $R = \mathbb{Z}[\sqrt{-3}]$ 及 $q = (1-\sqrt{-3})/2 = 2/(1+\sqrt{-3}) \in F$. 证明: I_q 不是主理想.

Let R be an integral domain and F be its quotient field. For any element $q \in F$, define $I_q := \{r \in R \,|\, rq \in R\}$.

(1) Show that each I_q is a nonzero ideal of R.

(2) Now suppose that $R = \mathbb{Z}[\sqrt{-3}]$ and let $q = (1-\sqrt{-3})/2 = 2/(1+\sqrt{-3}) \in F$. Show that I_q is not a principal ideal.

解答题四 (15 分) 记 $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ 是由常数项为整数的有理系数多项式构成的集合.

(1) 证明: R 是一个整环, 且它的可逆元只有 ± 1.

(2) 证明: R 中的不可约元恰为

- $\pm p$ (对所有素数 p),
- 常数项为 ± 1 的且在 $\mathbb{Q}[x]$ 中不可约的多项式 $f(x)$.

证明这些不可约元都是 R 中的素元.

(3) 证明 x 不可以被写成 R 中不可约元的乘积, 从而证明 R 不是唯一分解整环.

Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the set of polynomials in x with rational coefficients whose constant term is an integer.

(1) Prove that R is an integral domain and its units are ± 1.

(2) Show that the irreducibles in R are $\pm p$ where p is a prime in $\mathbb{Z}$ and the polynomials $f(x)$ that are irreducible in $\mathbb{Q}[x]$ and have constant term ± 1. Prove that these irreducibles are prime in R.

(3) Show that x cannot be written as a product of irreducibles in R and conclude that R is not a U.F.D.

解答题五 (15 分) 设 H 是 G 的子群, 令

$$K := \bigcap_{g \in G} gHg^{-1}$$

为群 H 所有共轭的交.

(1) 证明: K 是 G 的正规子群.

(2) 证明: 若 $[G:H]$ 是有限的, 则 $[G:K]$ 也是有限的.

Let H be a subgroup of G. Define

$$K := \bigcap_{g \in G} gHg^{-1}$$

to be the intersection of all conjugates of H.

(1) Show that K is a normal subgroup of G.

(2) Show that if $[G : H]$ is finite, then $[G : K]$ is finite. (Hint: first show that the intersection above defining K is essentially a finite intersection.)

解答题六 (15 分) 设 R 为一交换环. 一个导数算子是指一个映射 $D : R \to R$ 满足对所有 $a, b \in R$: $D(a+b) = D(a) + D(b)$ 和 $D(ab) = aD(b) + D(a)b$.

(1) 考虑环 $R[x]/(x^2)$, 证明: 存在一个双射

$$\big\{\text{导数算子 } D : R \to R\big\} \longleftrightarrow \big\{\text{环同态 } \varphi : R \to R[x]/(x^2) \text{ 使得 } \varphi \bmod x \text{ 是恒同}\big\}.$$

(2) 如果 D 是 R 上的一个导数算子且 $e \in R$ 是一个幂等元 (即 $e = e^2$), 证明: $D(e) = 0$.

Let R be a commutative ring. A *derivation* $D : R \to R$ is a map satisfying $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.

(1) Consider the ring $R[x]/(x^2)$, show that there is a bijection

$$\big\{\text{Derivations } D : R \to R\big\} \longleftrightarrow \left\{\begin{array}{l}\text{Ring homomorphisms } \varphi : R \to R[x]/(x^2) \\ \text{ such that } \varphi \bmod x = \mathrm{id}\end{array}\right\}.$$

(2) If D is a derivation of R and $e \in R$ is an idempotent (i.e. $e = e^2$), prove that $D(e) = 0$.

解答题七 (15 分) 令 p 为一奇素数. 设 G 是一个阶为 $p(p+1)$ 的有限群, 且假设 G 没有正规的西罗-p 子群.

(1) 求 G 中阶不为 p 的元素的个数.

(2) 证明: G 中阶不整除 p 的元素构成一个共轭类.

(3) 证明: $p+1$ 是 2 的幂.

Let p be an odd prime number, and let G be a finite group of order $p(p+1)$. Assume that G does not have a normal Sylow p-subgroup.

(1) Find the number of elements of G with order different from p.

(2) Show that the set of elements of G whose order does not divide p form exactly one conjugacy class.

(3) Prove that $p+1$ is a power of 2.

附加题一 (+5 分) 设 $K \subseteq H$ 为群 G 的子群满足 $K \lhd H$.

(1) 证明: H 在共轭作用下保持 $C_G(K)$ 不动 ($C_G(K)$ 是 K 在 G 中的中心化子).

(2) 设 $H \rhd G$ 和 $C_H(K) = 1$, 证明: H 与 $C_G(K)$ 交换.

Let G be a group and let $K \subseteq H$ be subgroups of G with $K \lhd H$.

(1) Prove that H normalizes $C_G(K)$ (the centralizer of K in G).

(2) If $H \lhd G$ and $C_H(K) = 1$, prove that H centralizes $C_G(K)$.

附加题二 (+5 分) 设 G 是一个有限群, 记 $\mathrm{Syl}_p(G)$ 为它的西罗 p-子群的集合.

(1) 如果 S 和 T 是 $\mathrm{Syl}_p(G)$ 中不同的元素使得 $\#(S\cap T)$ 取得最大值. 证明: $N_G(S\cap T)$ 没有正规的西罗 p-子群.

(2) 证明: $S\cap T=1$ 对所有 $S,T\in\mathrm{Syl}_p(G)$ $(S\neq T)$ 成立当且仅当对任一 G 的非平凡 p-子群 P, $N_G(P)$ 包含一个正规西罗 p-子群.

Let G be a finite group and let $\mathrm{Syl}_p(G)$ denote its set of Sylow p-subgroups.

(1) Suppose that S and T are distinct members of $\mathrm{Syl}_p(G)$ chosen so that $\#(S\cap T)$ is maximal among all such intersections. Prove that the normalizer $N_G(S\cap T)$ does not admit normal Sylow p-subgroup.

(2) Show that $S\cap T=1$ for all $S,T\in\mathrm{Syl}_p(G)$, with $S\neq T$, if and only if $N_G(P)$ has exactly one Sylow p-subgroup for every nonidentity p-subgroup P of G.