2022 秋: 代数学一 (实验班) 期中考试

姓名: ____________ 院系: ____________ 学号: ____________ 分数:

时间：110 分钟 满分：110 分, 总分不超过 100 分

判断题 在下表中填写 T 或 F (10 分)

1	2	3	4	5	6	7	8	9	10
T	F	F	F	F	T	T	F	T	F

1. 任意两个群 G 和 G' 之间都存在一个同态 $\phi : G \to G'$.

For any two groups G and G', there exists a homomorphism $\phi : G \to G'$.

True. There is always the trivial homomorphism $\phi : G \to G'$.

2. 用 $\mathbb{R}$ 表示所有实数构成的加法群, 取一个正整数 n, 记 $n\mathbb{R} = \{nr \,|\, r \in \mathbb{R}\}$. 那么 $\mathbb{R}/n\mathbb{R}$ 是一个 n 阶的循环群.

Let $\mathbb{R}$ denote the group of real numbers, n a positive integer, and put $n\mathbb{R} = \{nr \,|\, r \in \mathbb{R}\}$. Then $\mathbb{R}/n\mathbb{R}$ is a cyclic group of order n.

False. $n\mathbb{R}$ is in fact the entire $\mathbb{R}$ as every element in $\mathbb{R}$ is divisible by n. So $\mathbb{R}/n\mathbb{R}$ is trivial.

3. S_9 中存在一个元素的阶恰好是 18.

S_9 contains an element of order exactly 18.

False. If we want an element of order exactly 18, then we need a cycle of length at least 9 and we have no place to put the 2-cycle.

4. 如果 G 的交换子群 (或导出子群) 是它自己, 那么 G 是单群.

If the commutator subgroup of a group G is G itself, then G is a simple group.

False. If G is the direct product of two simple non-commutative groups H_1 and H_2, then $[G, G] = G$.

5. 如果 H 是群 G 的正规子群且 H' 是群 G' 的正规子群, 假设 H 同构于 H' 且 G 同构于 G'. 那么 G/H 同构于 G'/H'.

If H is a normal subgroup of G and H' is a normal subgroup of G', and suppose that H is isomorphic to H' and G is isomorphic to G', then G/H is isomorphic to G'/H'.

False. For a counterexample, $\mathbb{Z}/4\mathbb{Z}$ is clearly not isomorphic to $\mathbb{Z}/2\mathbb{Z}$, yet all groups $\mathbb{Z}$, $2\mathbb{Z}$, $4\mathbb{Z}$ are isomorphic. If we wanted to get $G/H \simeq G'/H'$, we need the isomorphism $G \xrightarrow{\simeq} G'$ to take H isomorphically to H'.

6. 一个有限幂零群是它所有西罗子群 (对不同的素数) 的直积.

A finite nilpotent group is the direct product of its Sylow subgroups (of different primes).

True. This is a theorem from the book.

7. 每一个阶为素数幂的群是可解的.

Every group of prime-power order is solvable.

True. Every group of prime-power order is nilpotent and in particular solvable.

8. 设 p 为一个素数, P 是一个有限群 G 的西罗 p-子群. 那么, 对 G 的任一子群 H, $H \cap P$ 是 H 的西罗 p-子群.

Let p be a prime number and P a Sylow p-subgroup of a finite group G. Then for any subgroup H of G, $H \cap P$ is a Sylow p-subgroup of H.

False. We need H to be normal for this to be correct. For example, if in a group G with more than one Sylow p-subgroup, then take H to be one Sylow p-subgroup and P another Sylow p-subgroup. Then $H \cap P$ is a proper subgroup of H, which cannot be a Sylow p-subgroup of H.

9. 设 G 是一个有限交换群. 则 G 的每个有限维不可约表示都是一维的.

Let G be a finite abelian group. Every finite dimensional irreducible representation of G is one-dimensional.

True. This is an exercise from the course.

10. 一个有限群 G 在一个有限集 X 上传递地作用. 则在 $\mathbb{C}[X] = \left\{ \sum_{x \in X} a_x[x] \,\middle|\, a_x \in \mathbb{C} \right\}$ 上诱导的 G 的表示是不可约的.

Let G be a finite group acting transitively on a finite set X. The induced representation of G on $\mathbb{C}[X] = \left\{ \sum_{x \in X} a_x[x] \,\middle|\, a_x \in \mathbb{C} \right\}$ is irreducible.

False. The space $\mathbb{C}[X]$ is clearly not irreducible, as it contains the subspace $\mathbb{C} \cdot \sum_{x \in X}[x]$.

解答题一 (15 分) 证明: 阶为 175 的群一定是交换群. 给出所有 (互不同构的) 阶为 175 的群. (如果使用素数平方阶群是交换的这样的结论，请证明.)

Prove that a group of order 175 must be commutative. List all groups of order 175, up to isomorphisms. (If you need to use a statement that a group of prime square order is abelian, you need to provide a proof.)

证明. $175 = 7 \times 5^2$. Let G be a group of order 175.

We first analyze the Sylow 5-subgroups. Let n_5 be the number of such groups. Then Sylow's theorem implies that $n_5 \equiv 1 \bmod 5$ and $n_5 \,|\, 7$. We deduce that $n_5 = 1$. Thus the Sylow 5-subgroup P_5 is a normal subgroup.

Next, we consider the Sylow 7-subgroups. Let n_7 be the number of such groups. Then $n_7 \equiv 1 \bmod 7$ and $n_7 \,|\, 25$. We have $n_7 = 1$. Thus the Sylow 7-subgroup P_7 is a normal subgroup.

Yet $P_5 \cap P_7 = \{1\}$. We have

$$G = P_5 \times P_7.$$

Next, we show that P_5 (with order 25) is commutative. Suppose that P_5 is not commutative, then $Z(P_5)$ has order 5 (as the center of a 5-group is non trivial.) Let σ be a generator of $Z(P_5)$, and let τ be an element of $P_5 \backslash Z(P_5)$. If τ has order 25, then $P_5 \simeq \mathbf{Z}_{25}$ is commutative; contradiction! If τ has order 5, then σ and τ generate a subgroup of P_5 isomorphic to $\mathbf{Z}_5 \times \mathbf{Z}_5$ as σ commutes with τ. By studying the order, $P_5 \simeq \mathbf{Z}_5 \times \mathbf{Z}_5$ is commutative; contradiction! So P_5 is commutative.

To sum up, G is an abelian group of order 175. By classification of finitely generated abelian group, such group is isomorphic to

$$\text{either } \mathbf{Z}_{175} \text{ or } \mathbf{Z}_5 \times \mathbf{Z}_{35}.$$

□

解答题二 (15 分)

设 (ρ, V) 是一个有限群 G 的有限维 $\mathbb{C}$-表示. 考虑其中 G-不变子空间

$$V^G := \{v \in V \,|\, \rho(g)(v) = v \text{ for all } g \in G\}.$$

(1) 证明: $\dim V^G$ 等于平凡表示在 V 中的重数.

(2) 证明: $\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$.

(3) 请用 $\rho(g)$ $(g \in G)$ 的线性组合构造一个满射 $\phi : V \to V^G$, 使得 $\phi^2 = \phi$ (即 ϕ 是一个投影) 且 ϕ 是表示同态.

Let (ρ, V) be a finite dimensional $\mathbb{C}$-representation of a finite group G. Consider the G-invariant subspace

$$V^G := \{v \in V \,|\, \rho(g)(v) = v \text{ for all } g \in G\}.$$

(1) Show that $\dim V^G$ is the same as the multiplicity of the trivial representation appearing in V.

(2) Show that $\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$.

(3) Construct a *surjective* map $\phi : V \to V^G$, expressed in terms of a linear combination of linear operators $\rho(g)$ for $g \in G$, such that $\phi^2 = \phi$ (i.e. ϕ is a projection) and ϕ is a homomorphism.

证明. (1) Write V as a direct sum of irreducible subrepresentations:

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

Then $V^G = W_1^G \oplus W_2^G \oplus \cdots \oplus W_r^G$. But W_i^G is always a subrepresentation of W_i. If some W_i is irreducible and nontrivial, W_i^G must be trivial. Yet if some W_i is trivial, $W_i^G = W_i$.

To sum up, we have V^G is the direct sum of all trivial factors of V, and thus $\dim V^G$ is the same as the multiplicity of trivial representation in V.

(2) By character formula, the multiplicity of trivial representation in V is

$$\langle V, \mathbf{1} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g).$$

(3) Consider the homomorphism

$$\phi = \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

For each $v \in V$ and $h \in G$,

$$\rho(h)\phi(v) = \rho(h)\Big(\frac{1}{|G|} \sum_{g \in G} \rho(g)(v)\Big) = \frac{1}{|G|} \sum_{g \in G} \rho(hg)(v) = \frac{1}{|G|} \sum_{k \in G} \rho(k)(v) = \phi(v).$$

So $\phi(v) \in V^G$. Yet, for $v \in V^G$, we have

$$\phi(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(v) = \frac{1}{|G|}\sum_{g\in G} v = v.$$

Thus ϕ restricted to V^G is the identity. In particular, this says that ϕ is surjective and $\phi^2 = \phi$.

Finally, we check that ϕ is a homomorphism, i.e. for $h \in G$,

$$\phi \circ \rho(h) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\rho(h) = \frac{1}{|G|}\sum_{g\in G}\rho(gh) = \frac{1}{|G|}\sum_{k\in G}\rho(hk) = \rho(h)\circ\phi,$$

where the change of variable is that $k = h^{-1}gh$. □

解答题三 (15 分) (1) 设 G 是一个群. 证明如下两个集合之间有一一对应:

(a) G 中指数为 2 的子群 H,

(b) 非平凡的同态 $\phi : G \to \mathbf{Z}_2$.

(2) 对正整数 $n \geq 3$, 给出二面体群 D_{2n} 中所有指数为 2 的子群 (用生成元表出). 证明你的结论.

(1) Let G be a group. Show that there is a bijection between

(a) subgroups H of G of index 2; and

(b) nontrivial homomorphism $\phi : G \to \mathbf{Z}_2$.

(2) Let $n \geq 3$ be a positive integer. Describe all subgroups of the dihedral group D_{2n} of index 2, by providing their generators. Justify your answers.

证明. (1) Given a subgroup H of G of index 2, it must be normal. Then we have a natural quotient homomorphism

$$\phi : G \to G/H \simeq \mathbf{Z}_2.$$

Conversely, given a nontrivial homomorphism $\phi : G \to \mathbf{Z}_2$, its kernel H is a subgroup of G of index 2.

It is clear that the two maps are inverse of each other.

(2) We use (1) to find subgroups of $D_{2n} = \langle r, s \,|\, r^n = s^2 = 1, srs = r^{-1} \rangle$ of index 2.

When n is odd, to give a homomorphism $\phi : D_{2n} \to \mathbf{Z}_2$, we must have

$$n\phi(r) = \phi(r^n) = 0$$

Thus $\phi(r) = 0$. To get a nontrivial homomorphism, we are forced to take $\phi(s) = 1$. It is easy to verify that $\phi(s^2) = 0$ and $\phi(srs) = \phi(r^{-1})$. This defines a homomorphism $\phi : D_{2n} \to \mathbf{Z}_2$. Its kernel is precisely all elements of the form r^i for some i, namely $\langle r \rangle$. So in this case, D_{2n} has a unique subgroup of index 2, namely $\langle r \rangle$.

When n is even, we want to find all homomorphisms $\phi : D_{2n} \to \mathbf{Z}_2$. If $\phi(r) = 0$, we may argue as above to see that $\phi(s) = 1$ and $\ker \phi = \langle r \rangle$ is a subgroup of D_{2n} of index 2. If $\phi(r) = 1$, we can check that for either $\phi(s) = 0$ or 1, the condition

$$\phi(r^n) = \phi(s^2) = 0 \quad \text{and} \quad \phi(srs) = \phi(r^{-1})$$

holds. So either case gives a homomorphism $\phi : D_{2n} \to \mathbf{Z}_2$. In the case when $\phi(s) = 0$, $\phi(r^i s^j) = i$, and thus $\ker \phi = \langle r^2, s \rangle$ is a subgroup of D_{2n} of index 2. In the case when $\phi(s) = 1$, $\phi(r^i s^j) = i + j$, and thus $\ker \phi = \langle r^2, rs \rangle$ is a subgroup of D_{2n} of index 2. □

解答题四 (15 分) 设 G 是一个有限群, H 是 G 的真子群 (即 $H \lneq G$). 证明: 并集 $\bigcup_{g\in G} gHg^{-1}$ 不是整个的群 G.

For G a finite group and H a proper subgroup (i.e. $H \lneq G$). Show that the union $\bigcup_{g\in G} gHg^{-1}$ cannot be equal to the entire G.

证明. We simply note that for any two elements $g_1, g_2 \in G$, if $g_1 = g_2 h$, then

$$g_1 H g_1^{-1} = g_2 h H h^{-1} g_2^{-1} = g_2 H g_2^{-1}$$

are the same set. Thus, when taking the union $\bigcup_{g\in G} gHg^{-1}$, it is enough to take the union over all representatives of the cosets G/H. There are $|G|/|H|$ such representatives, yet each set gHg^{-1} has size equal to $|H|$. So totally, in the union $\bigcup_{g\in G} gHg^{-1}$ (counting multiplicity) there are $|G|$ elements. Clearly, the element 1 belongs to each of gHg^{-1}. So the union is not disjoint. So the total union has strictly less that $|G|$ elements, and thus cannot be equal to the entire G. □

解答题五 (15 分) 假设群 G 在集合 X (可能是无限集) 上作用, H 是群 G 中指数有限的子群. 对 $x \in X$, 用 H_x 和 G_x 分别表示群 H 和 G 在 x 处的稳定子群.

(1) 证明: H 在 X 上有有限个轨道.

(2) 证明: 如果群 H 在 X 上的作用是传递的,且对某 $x \in X$ 有 $H_x = G_x$, 则 $H = G$.

(3) 证明: 如果 H 是一个正规子群, 则指数 $[G_x : H_x]$ (不管有限与否) 不依赖于 x 的选取.

Suppose that G is a group acting transitively on a set X (which may be infinite) and that H is a finite index subgroup of G. For $x \in X$, write H_x and G_x for its stabilizers in H and G, respectively.

(1) Show that H has finitely many orbits on X.

(2) Show that, if the action of H on X is transitive and for some $x \in X$, $H_x = G_x$; then H is all of G.

(3) Show that if H is normal, then $[G_x : H_x]$ (finite or not) is independent of x.

证明. (1) Write G as the union of right cosets of H: $G = Hg_1 \sqcup Hg_2 \sqcup \cdots \sqcup Hg_r$ for some $g_1, \ldots, g_r \in G$ and $r = [G : H]$. Fix $x \in X$. We show that every point $x' \in X$ is in the same H-orbit of at least one of $\{g_1x, g_2x, \ldots, g_rx\}$. Indeed, since G acts transitively on X, $x' = g \cdot x$ for some $g \in G$. In the coset decomposition, $g = hg_i$ for some $i \in \{1, \ldots, r\}$ and $h \in H$. Thus

$$x' = gx = hg_ix$$

lies in the same H-orbit of g_ix. So there are only finitely many H-orbits on X.

(2) We keep the notation as in (1) and assume that x is the chosen point. Suppose that $r > 1$ and hence we may assume that $g_2 \notin H$. Consider the point $g_2x \in X$. By the transitivity of the action of H, $g_2x = hx$ for some $h \in H$. Thus, $h^{-1}g_2x = x$. Thus, $h^{-1}g_2 \in G_x = H_x$. This in particular implies that $g_2 \in H$, contradicting our earlier assumption. So $H = G$.

(3) Once again, keep the notation as in (1). For $x' = gx$ for some $g \in G$, we note that $G_{x'} = gG_xg^{-1}$; indeed,

$$h \in G_{x'} \Leftrightarrow hx' = x' \Leftrightarrow hgx = gx \Leftrightarrow g^{-1}hgx = x \Leftrightarrow g^{-1}hg \in G_x \Leftrightarrow h \in gG_xg^{-1}.$$

Similarly, as H is normal,

$$H_{x'} = gG_xg^{-1} \cap H = gG_xg^{-1} \cap gHg^{-1} = g(G_x \cap H)g^{-1} = gH_xg^{-1}.$$

There is obviously a one-to-one correspondence between G_x/H_x and gG_xg^{-1}/gH_xg^{-1}, sending aH_x to $gag^{-1} \cdot gH_xg^{-1}$. In particular, $[G_x : H_x] = [G_{x'} : H_{x'}]$ and therefore, $[G_x : H_x]$ is independent of x. □

解答题六 (10 分)

一个群 G 在集合 X 上的作用称为*双传递的*, 如果这个作用是传递的，且 G 在 $X\times X-\Delta$ 上的作用是传递的, 这里 $\Delta\subset X\times X$ 是对角线集合 (即对 $x_1,y_1,x_2,y_2\in X$ ($x_1\neq y_1$, $x_2\neq y_2$), 存在元素 $g\in G$ 使得 $gx_1=x_2$ 且 $gy_1=y_2$). 设 p 是一个素数, 记 $G=\mathrm{GL}_2(\mathbb{F}_p)$.

(1) 给出 G 的一个西罗 p-子群, 并计算它的正规化子.

(2) 证明: G 有 $p+1$ 个不同的西罗 p-子群.

(3) 证明: G 在所有西罗 p-子群构成的集合 X 上的作用是双传递的.

Recall that a permutation action of a group G on a set X is *doubly transitive* if the action on X is transitive and the action on $X\times X-\Delta$ is transitive where $\Delta\subset X\times X$ is the diagonal (i.e., for $x_1,y_1,x_2,y_2\in X$ with $x_1\neq y_1$ and $x_2\neq y_2$ there exists $g\in G$ such that $gx_1=x_2$ and $gy_1=y_2$). Let p be a prime number and let $G=\mathrm{GL}_2(\mathbb{F}_p)$.

(1) Find a Sylow p-subgroup of G and compute its normalizer.

(2) Show that G has $p+1$ distinct Sylow p-subgroups.

(3) Show the action of G on the set X of Sylow p-subgroups is doubly transitive.

证明. (1) We know that $|\mathrm{GL}_2(\mathbb{F}_p)|=(p^2-1)(p^2-p)$. So a Sylow p-subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ has order p. For example,

$$N=\left\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\,\middle|\,n\in\mathbb{F}_p\right\}.$$

We compute its normalizer: for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in N_G(N)$, we need (at least)

$$\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^{-1}\in N$$

or equivalently, for some $m\in\mathbb{F}_p$,

$$\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\quad\text{i.e.}\quad\left(\begin{smallmatrix}a&a+b\\c&c+d\end{smallmatrix}\right)=\left(\begin{smallmatrix}a+mc&b+md\\c&d\end{smallmatrix}\right).$$

By looking at the $(2,2)$-entry, we see that $c=0$. On the other hand, for

$$B=\left\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\,\middle|\,a,d\in\mathbb{F}_p^\times,\,b\in\mathbb{F}_p\right\}$$

it is clear that B normalize N, forcing $N_G(N)=B$.

(2) As all Sylow p-subgroups of G are conjugate, so the set of Sylow p-subgroups can be identified with $G/N_G(N)=G/B$, which is of size $\dfrac{(p^2-1)(p^2-p)}{(p-1)^2p}=p+1$.

(3) Sylow's theorem shows that G acts on X transitively. So clearly, Δ is an orbit of the G-action. It is enough to show that G acts on $X\times X\backslash\Delta$ transitively. For this, it is enough to compute the stabilizer of some pair of Sylow subgroups. We consider $N=\left(\begin{smallmatrix}1&\mathbb{F}_p\\0&1\end{smallmatrix}\right)$ and $N^{\mathrm{op}}=\left(\begin{smallmatrix}1&0\\\mathbb{F}_p&1\end{smallmatrix}\right)$ (both of them have cardinality p, so a Sylow subgroups). The stabilizer

of the pair (N, N^{op}) is the intersection

$$N_G(N) \cap N_G(N^{\mathrm{op}}) = B \cap B^{\mathrm{op}} = \left(\begin{smallmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{smallmatrix}\right) =: T,$$

where $B^{\mathrm{op}} = \left(\begin{smallmatrix} \mathbb{F}_p^\times & \mathbb{F}_p \\ 0 & \mathbb{F}_p^\times \end{smallmatrix}\right)$. From this we see that the orbit containing the pair (B, B^{op}) is

$$|G|/|T| = \frac{(p^2-1)(p^2-p)}{(p-1)^2} = p(p+1) = |X \times X - \Delta|.$$

From this, we see that $X \times X - \Delta$ is one orbit under G-action and thus G acts doubly transitively on X. $\square$

解答题七 (10 分)

设 G 是一个阶为 n 的有限群. 则左平移定义了一个同态 $\pi : G \to S_n$: 对 $g \in G$, 对应的在 G 上的置换为 $\pi_g(x) = gx$ $(x \in X)$.

(1) 证明: π_g 是一个奇置换当且仅当 g 的阶是偶数且 $[G : \langle g\rangle]$ 是奇数.

(2) 证明: 如果 G 的一个西罗 2-子群非平凡且是循环群, 则 G 有一个指数为 2 的子群.

Let G be a finite group of order n. There is a homomorphism $\pi : G \to S_n$, where $g \in G$ maps to the permutation π_g: for any $x \in G$, $\pi_g(x) = gx$.

(1) Show that π_g is an odd permutation if and only if g has even order and $[G : \langle g\rangle]$ is odd.

(2) Show that if a Sylow 2-subgroup of G is nontrivial and cyclic, then G has a subgroup H with $[G : H] = 2$.

证明. (1) Let r be the order of g. Then for any $x \in G$, the permutation π_g takes

$$x \xmapsto{\pi_g} gx \xmapsto{\pi_g} g^2x \xmapsto{\pi_g} \cdots \xmapsto{\pi_g} g^r x = x,$$

and it is clear that for any $i \in \{1, \dots, r-1\}$, $g^i x \neq x$. Thus π_g breaks up G into disjoint union of $[G : \langle g\rangle]$ cycles, each is a r-cycle.

But we know that r-cycle is the product of $r-1$ transpositions; so for π_g to be an odd permutation, we need and only need $(r-1) \cdot [G : \langle g\rangle]$ to be an odd number, i.e. r is even and $[G : \langle g\rangle]$ is odd.

(2) Consider the homomorphism $\pi : G \to \S_n$ given by left translation action of G on itself. There is a natural homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\}$ sending even permutations to 1 and odd permutation to -1. The composition is

$$(\star) \qquad G \xrightarrow{\pi} S_n \xrightarrow{\mathrm{sgn}} \{\pm 1\}.$$

We need to show that the composition is surjective, then the kernel would give a subgroup of G of index 2.

For this, we need to show that for some $g \in G$, π_g is an odd permutation. Since the Sylow 2-subgroup P_2 of G is nontrivial and cyclic; let σ_2 be its generator. Then σ_2 has even order and $[G : \langle \sigma_2 \rangle] = [G : P_2]$ is odd. By (1), π_σ is an odd permutation and hence $(\star)$ is surjective. We are done. $\square$

解答题八 (5 分)

证明: 如果群 G 的中心是平凡的, 那么它的自同构群 $\mathrm{Aut}(G)$ 的中心也是平凡的.

Let G be a group. Show that if G has trivial center, then its automorphism group $\mathrm{Aut}(G)$ has trivial center.

证明. If $\psi \in Z(\mathrm{Aut}(G))$ is a nontrivial element in the center of the automorphism group of G. In particular, ψ must commute with any automorphism induced by conjugation by an element of G. Namely, as automorphisms of G, for each $g \in G$, we have

$$\mathrm{Ad}_g \circ \psi = \psi \circ \mathrm{Ad}_g.$$

Applying this to an element $h \in G$, we deduce that

$$g\psi(h)g^{-1} = \psi(ghg^{-1}) = \psi(g)\psi(h)\psi(g)^{-1}$$

for any $g, h \in G$. (Last equality is because ψ is a homomorphism.)

Rearranging terms, we deduce that

$$\psi(h)g^{-1}\psi(g) = g^{-1}\psi(g)\psi(h).$$

This means that $g^{-1}\psi(g)$ comes with any element $\psi(h)$ and hence $g^{-1}\psi(g) \in Z(G) = \{1\}$. Thus $\psi(g) = g$. $\square$