Topics in Number theory: Special values of L-functions
Exercise 1 (due on September 26)

Choose 4 out of 8 problems to submit, must including Exercise 1.5 (The problems are chronically ordered by the materials, not necessarily by difficulties. I do recommend read all problems.)

Exercise 1.1 (Gauss sums)**.** Let $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character of order $N \geq 2$, we define the Gauss sum of η as follows:

$$G(\eta) := \sum_{a=1}^{N-1} \eta(a) e^{2\pi i \cdot a/N} \in \mathbb{C}. \tag{1.1.1}$$

Prove the following properties of the Gauss sum.

(1) If η' is a Dirichlet character of order N' with $(N, N') = 1$, then $\eta\eta'$ may be viewed as a Dirichlet character of order NN'. Show that in this case $G(\eta\eta') = \eta(N')\eta'(N)G(\eta)G(\eta')$.

(2) If η is primitive, then $|G(\eta)| = \sqrt{N}$.

(3) When η and η' are both Dirichlet characters of same order N such that $\eta\eta'$ is a primitive Dirichlet character of order N, show that

$$G(\eta\eta') = \frac{G(\eta)G(\eta')}{J(\eta, \eta')}, \tag{1.1.2}$$

where $J(\eta, \eta')$ is the Jacobi sum

$$J(\eta, \eta') := \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \eta(a)\eta'(1-a),$$

where we use the convention that $\eta(a) = 0$ if $(a, N) \neq 1$.

Remark 1. It would be interesting to compare Gauss sums with the Gamma functions. In some sense, the definition of (1.1.1) may be viewed as an integral of the product of an additive character $e^{2\pi i(\cdot)/N}$ of $\mathbb{Z}/N\mathbb{Z}$ and a multiplicative character η of $(\mathbb{Z}/N\mathbb{Z})^\times$. Similarly, the definition of Gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

can also be viewed as an integral of the product of the additive character e^{-t} and the multiplicative character t^s.

Analogous to the relation (1.1.2) between Gauss sum and the (finite) Jacobi sum, Gamma functions satisfy a similar property:

$$B(s, s') = \frac{\Gamma(s)\Gamma(s')}{\Gamma(s+s')},$$

where $B(s, s')$ is a beta function

$$B(s, s') = \int_0^1 t^{s-1}(1-t)^{s'-1} dt.$$

Exercise 1.2. (Modified Mahler basis) In this problem, we give a different orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$. Consider the function $f(z) = \frac{z^p - z}{p}$ on $\mathbb{Z}_p$.

(1) Show that $f \in \mathcal{C}^\circ(\mathbb{Z}_p, \mathbb{Z}_p)$.

Consider the following inductively defined functions:

$$f^{\{0\}}(z) = z, \quad f^{\{1\}}(z) = f(z) = \frac{z^p - z}{p}, \quad f^{\{2\}}(z) = f^{\{1\}}\Big(\frac{z^p - z}{p}\Big) = \frac{\big(\frac{z^p-z}{p}\big)^p - \frac{z^p-z}{p}}{p},$$

$$f^{\{k+1\}}(z) = f\big(f^{\{k\}}(z)\big), \qquad \text{for } k \geq 1.$$

For $n \geq 0$, write $n = n_0 + n_1 p + n_2 p^2 + \cdots$ for the p-adic expansion of n, i.e. each $a_i \in \{0, 1, \dots, p-1\}$, put

$$e_n(z) = \big(f^{\{0\}}(z)\big)^{n_0}\big(f^{\{1\}}(z)\big)^{n_1}\big(f^{\{2\}}(z)\big)^{n_2}\cdots$$

We call $\{e_n(z)\}$ a *modified Mahler basis.*

(2) Prove that $e_p(z) + \binom{z}{p} \in \mathbb{Z}_p[z]$.
(3) Prove that each $e_n(z)$ may be written as a $\mathbb{Z}_p$-linear combination of binomial functions $\binom{z}{m}$'s, and show that the change of basis matrix from the Mahler basis to $e_n(z)$ is upper triangular with all entries in $\mathbb{Z}_p$ and diagonal entries in $\mathbb{Z}_p^\times$.
(4) Deduce that $\{e_n(z) \,|\, n \geq 0\}$ form an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$.
(5) Assume that $p \geq 3$. Recall that $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times$, where μ_{p-1} is the subgroup of $(p-1)$th roots of unity in $\mathbb{Q}_p$. The group μ_{p-1} acts naturally on $\mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$ such that for $\zeta \in \mu_{p-1}$, it sends $h(z)$ to $h(\zeta z)$. Show that each of $e_n(z)$ is an eigenfunction for this action.

Remark 2. We call $e_n(z)$'s the *modified Mahler basis.* As (2) suggested, $e_n(z)$ is essentially the "leading terms" of $\binom{z}{n}$ up to a constant multiple.

The disadvantage of modified Mahler basis is that it is not compatible with the Amice transform. However, part (5) shows that the modified Mahler basis are formed by μ_{p-1}-eigenfunctions, which are useful in some applications.

Exercise 1.3. (Orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$) Let $\mathbb{Q}_{p^r}$ be the unramified extension of $\mathbb{Q}_p$ of degree r, and $\mathbb{Z}_{p^r}$ be its ring of integers. In this problem, we will produce an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$ that is similar to the modified Mahler basis defined in the previous problem.

Let σ denote the (arithmetic) Frobenius on $\mathbb{Z}_{p^r}$, i.e. the automorphism of $\mathbb{Z}_{p^r}$ whose reduction modulo p sends $\bar{x}$ to $\bar{x}^p$. Write $z_0 : \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^r}$ for the identify function, i.e. $z_0(a) = a$. We then inductively define

$$z_{j+1}(a) = \sigma(z_j(a)) \qquad \text{for } j \geq 0.$$

Clearly, $z_{j+r} = z_j$ for $j \geq 0$. It is also clear that $\mathbb{Q}_{p^r}[z_0, \dots, z_{r-1}]$ is a dense subring of $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Q}_{p^r})$ (but $\mathbb{Z}_p[z_0, \dots, z_{r-1}]$ is not dense in $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$).

We define inductively

$$f_0 := 1, \quad f_1 := z_0, \quad f_p := \frac{z_0^p - z_1}{p}, \quad f_{p^{i+1}} = f_p \circ f_{p^r} = \frac{f_{p^i}^p - \sigma(f_{p^i})}{p}, \text{ with } i = 1, 2, \dots$$

For example, $f_{p^2} = \dfrac{\left(\frac{z_0^p - z_1}{p}\right)^p - \frac{z_1^p - z_2}{p}}{p}$.

If $m = s_0 + ps_1 + p^2 s_2 + \cdots$ is the p-adic expansion of a positive integer (with $s_i \in \{0, \dots, p-1\}$), we set

$$f_m := f_1^{s_0} f_p^{s_1} f_{p^2}^{s_2} \cdots$$

Finally, if $\mathbf{m} = (m_0, \dots, m_{r-1}) \in \mathbb{Z}_{\geq 0}^r$ is an r-tuple of index, we set

$$\mathbf{f_m} := f_{m_0} \cdot \varphi(f_{m_1}) \cdots \varphi^{r-1}(f_{m_{r-1}}). \tag{1.3.1}$$

(1) Show that each function $\mathbf{f_m}$ is a continuous function in $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$, and compute its leading coefficients, as a polynomial in $z_0, \dots, z_{r-1}$.

(2) Show that $\mathbf{f_m}$'s form an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$.

(Hint: it might be helpful to compare this to a "known" (noncanonical) Mahler basis: choose a $\mathbb{Z}_p$-linear isomorphism

$$c : \mathbb{Z}_{p^r} \xrightarrow{\quad\cong\quad} (\mathbb{Z}_p)^r$$
$$a \longmapsto (\mathsf{c}_0^*(a), \dots, \mathsf{c}_{r-1}^*(a)).$$

Here we may view each c_j^* as a function $\mathbb{Z}_{p^r}$ with values in $\mathbb{Z}_p$. Then the functions $\mathbf{u_m} : a \mapsto \binom{\mathsf{c}_0^*(a)}{m_0} \cdots \binom{\mathsf{c}_{r-1}^*(a)}{m_{r-1}}$ for $\mathbf{m} \in \mathbb{Z}_{\geq 0}^r$ form an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r})$ with respect to the maximal norm $|| \cdot ||$. It is then a question to compare the two bases $\mathbf{f_m}$ and $\mathbf{u_m}$.)

Exercise 1.4 (An explicit formula for ψ-operator)**.** Let p be a prime number. Recall that on $\mathbb{Z}_p[\![T]\!]$, we have defined an operator φ such that $\varphi(T) = (1+T)^p - 1$. There is a left inverse to φ, given as follows: each $F \in \mathbb{Z}_p[\![T]\!]$ can be written uniquely as $F = \sum_{i=0}^{p-1} (1+T)^i \varphi(F_i)$; then $\psi(F) = F_0$.

(1) Let ζ_p denote a primitive p-th root of unity. Prove that ψ-operator admits the following characterization: for $F \in \mathbb{Z}_p[\![T]\!]$, $\psi(F)$ is the unique power series in $\mathbb{Z}_p[\![T]\!]$ such that

$$\psi(F)((1+T)^p - 1) = \frac{1}{p} \sum_{i=0}^{p-1} F((1+T)\zeta_p^i - 1). \tag{1.4.1}$$

(2) Show that φ and ψ can be naturally extended to the p-adic completion of $\mathbb{Z}_p((T))$, denoted by $\mathbb{A}_{\mathbb{Q}_p}$.

(3) Show that $\psi\Big(\dfrac{1}{T}\Big) = \dfrac{1}{T}$. (One might find (1.4.1) useful, but there is a "better" proof without using it.)

Remark 3. (1) Without going into details, let us simply remark that the actions of φ, ψ, and $\Gamma \cong \mathbb{Z}_p^\times$ on $\mathbb{Z}_p[\![T]\!]$ and their extensions to $\mathbb{A}_{\mathbb{Q}_p}$ defines the most important ground ring for (φ, Γ)-modules; this is a very useful tool in studying p-adic Hodge theory of local fields. We may encounter more of these constructions in the future (if we decide to introduce Coleman's power series).

(2) The right hand side of formula (1.4.1) may be viewed as taking the trace from $\mathbb{Z}_p[\![T]\!]$ to $\varphi(\mathbb{Z}_p[\![T]\!])$.

Exercise 1.5 ("Miraculous congruence" encoded in p-adic L-functions)**.** Assume $p \geq 3$ for simplicity. We have constructed p-adic Dirichlet L-functions as p-adic measures on $\mathbb{Z}_p^\times$ that interpolates special values of (p-modified) Dirichlet L-functions. It is natural to ask: is the p-adic Dirichlet L-function uniquely determined by these interpolation values? In fact, the answer is that these values "overdetermine" the p-adic L-functions. (We will discuss this in lectures at a later stage.) Assume that $p \geq 3$ is an odd prime number.

(1) Let G be a general profinite group and let $\chi : G \to R^\times$ be a continuous p-adic character with values in a p-adically complete ring R, then it induces a continuous ring homomorphism $\tilde{\chi} : \mathbb{Z}_p[\![G]\!] \to R$. Alternatively, χ can be viewed as a R-valued function on G, so one can integrate against a p-adic measure on G.

Prove that we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p[\![G]\!] & \xrightarrow{\ \cong\ } & \mathcal{D}_0(G, \mathbb{Z}_p) \\ & \tilde{\eta} \searrow \qquad \swarrow \mu \mapsto \int_G \eta(g) d\mu(g) & \\ & R & \end{array}$$

(2) Write $\Delta := \mathbb{F}_p^\times$, which may be viewed as a subgroup of $\mathbb{Z}_p^\times$ via Teichmüller character ω. Give an canonical isomorphism $\Phi : \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![X]\!]$, so that $X = [\exp(p)] - 1$, where $\exp(p) = 1 + p + \frac{p^2}{2!} + \cdots$ is the formal expansion.

(3) Let $\eta : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ be a finite character and let $n \in \mathbb{Z}_{\geq 0}$; we may form the p-adic character

$$\begin{aligned} \chi_{\eta,n} : \mathbb{Z}_p^\times &\longrightarrow \overline{\mathbb{Q}}_p^\times \\ a &\longmapsto \eta(a)a^n. \end{aligned}$$

If we denote by $\bar{\chi}_{\eta,n}$ the restriction of $\chi_{\eta,n}$ to Δ, then for any $\mu \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p^\times} \eta(x)x^n d\mu(x) = \Phi(\mu)|_{\Delta = \bar{\chi}_{\eta,n},\, T = \chi_{\eta,n}(\exp(p)) - 1}.$$

(4) Prove that two p-adic measures $\mu_1, \mu_2 \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ are equal if for any $n \in \mathbb{Z}_{\geq 0}$,

$$\int_{\mathbb{Z}_p^\times} x^n d\mu_1(x) = \int_{\mathbb{Z}_p^\times} x^n d\mu_2(x).$$

(Hint: Show that the difference $\mu_1 - \mu_2$ is divisible by some infinite product.)

(5) Prove that two p-adic measures $\mu_1, \mu_2 \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ are equal if for a *fixed* $n \in \mathbb{Z}_{\geq 0}$ but for all finite characters $\eta : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ for all r, we have

$$\int_{\mathbb{Z}_p^\times} \eta(x)x^n d\mu_1(x) = \int_{\mathbb{Z}_p^\times} \eta(x)x^n d\mu_2(x).$$

Exercise 1.6. (Kubota–Leopoldt p-adic L-function) In the second and the third lectures, we have constructed the p-adic Dirichlet L-function when the (tame) Dirichlet character η is nontrivial. For the case when $\eta = \mathbf{1}$, we should also construct the corresponding p-adic zeta-function, traditionally called the Kubota–Leopoldt p-adic L-function. Unfortunately, this will not be a p-adic measure on $\mathbb{Z}_p^\times$, but only a "quasi-measure", which is philosophically related to that ζ-function has a pole at $s = 1$ (so should the p-adic zeta have). For this, we need some technical maneuver.

Pick $a \in \mathbb{Z}_{>1}$ prime to p. Consider

$$\zeta_a(s) := (1 - a^{1-s}) \cdot \zeta(s) = \sum_{\substack{n \geq 1 \\ (n,a)=1}} \frac{1}{n^s} - a \cdot \sum_{\substack{n \geq 1 \\ a \mid n}} \frac{1}{n^s}$$

$$A_a(T) = (1 - a\gamma_a)\Big(\frac{1+T}{1-(1+T)}\Big) = \frac{1+T}{1-(1+T)} - a \cdot \frac{(1+T)^a}{1-(1+T)^a},$$

where $\gamma_a \in \Gamma = \mathbb{Z}_p^\times$ is the element corresponds to $a \in \mathbb{Z}_p^\times$, which acts on $\mathbb{Z}_p[\![T]\!]$ by sending T to $(1+T)^a - 1$.

(1) Show that $A_a(T) \in \mathbb{Z}_p[\![T]\!]$ defines a p-adic measure; so is $A_a^{\{p\}}(T) := (1-\varphi\psi)(A_a(T))$.

Define $\mu_a^{\{p\}}$ to be the p-adic measure associated to $A_a^{\{p\}}(T)$ via Amice transform. For any primitive character $\eta : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{Q}^{\mathrm{alg},\times}$, define

$$L^{\{p\}}(\eta, s) = (1 - \eta(p)p^{-s}) \cdot L(\eta, s).$$

$$L_a^{\{p\}}(\eta, s) = (1 - a^{1-s}) \cdot L^{\{p\}}(\eta, s) = \sum_{\substack{n\geq 1\\(n,ap)=1}} \frac{1}{n^s} - a \cdot \sum_{\substack{n\geq 1\\(n,p)=1}} \frac{1}{(an)^s}$$

(2) Show that for any character η and any $n \in \mathbb{Z}_{\geq 0}$, we have

$$\int_{\mathbb{Z}_p^\times} \eta(x)x^n d\mu_a^{\{p\}}(x) = L^{\{p\}}(\eta, -n).$$

(3) Recall the identification $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![X]\!]$. We may define the *Kubota–Leopoldt p-adic L-function* to be the element

$$\mu_{\mathrm{KL}} := \frac{\mu_a^{\{p\}}}{(1 - a[\gamma_a])} \in \mathbb{Z}_p[\Delta] \otimes \frac{1}{X}\mathbb{Z}_p[\![X]\!].$$

Sometimes, this is called a *pseudo-measure*; show that μ_{KL} is independent of the choice of $a \in \mathbb{Z}_p^\times$. (Hint: We need only to prove that $(1-b\gamma_b)(\mu_a^{\{p\}}) = (1-a\gamma_a)(\mu_b^{\{p\}})$ for two different $a, b \in \mathbb{Z}_{>1}$ relatively prime to p. One can make use of Exercise 1.5(4)(5).)

Remark 4. Our definition of pseudo-measure slightly differs from that of Jacinto–Williams' note, who shifted the p-adic Kubota–Leopolds L-function so that the pole is at $s = 0$.

Exercise 1.7 (A more classical version of p-adic L-function)**.** Historically, there is also an old version of p-adic L-function which is really just p-adic functions. In this exercise, we recover the classical p-adic L-function from the p-adic measures, and we will see that the p-adic measures contains stronger congruence relations than classical p-adic L-functions.

(To avoid talking about pseudo-measures, we again work with p-adic Dirichlet L-functions.) Let η be a primitive Dirichlet character of conductor N (with $p \nmid N$). We have constructed a p-adic measure $\mu_\eta^{\{p\}}$ such that

$$\int_{\mathbb{Z}_p^\times} x^n d\mu_\eta^{\{p\}}(x) = L^{\{p\}}(\eta, -n).$$

(This measure also interpolates Dirichlet L-functions for varying the character at p; we will not use it here.)

We are interested in understanding the p-adic function $\zeta_{p,i}$ on $\mathbb{Z}_p$ for $i = 0, 1, \ldots, p-2$, defined by for $s \in \mathbb{Z}$ such that $s \equiv i \bmod p-1$,

$$\zeta_{p,i}(s) := \int_{\mathbb{Z}_p^\times} x^s d\mu_\eta^{\{p\}}(x) = L^{\{p\}}(\eta, -s).$$

(1) Show that $\zeta_{p,i}(s)$ extends naturally to a continuous function on $s \in \mathbb{Z}_p$. (So far, this is weaker than a function on $s \in \mathcal{O}_{\mathbb{C}_p}$.)

Now we study these functions $\zeta_{p,i}$ more carefully. Abstractly by Exercise 1.5, we may view $\mu_\eta^{\{p\}}$ as an element in $\mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![X]\!]$, where $X = [\exp(p)] - 1$. (Here we view $\Delta = \mathbb{F}_p^\times$ as a subgroup of $\mathbb{Z}_p^\times$ via the Teichmüller character ω.) For $i = 0, \dots, p-2$, write $\mu_{\eta,i}(X) \in \mathcal{O}[\![X]\!]$ for the image of $\mu_\eta^{\{p\}}$ under the map $\Delta \to \mathbb{Z}_p^\times$ sending x to $\omega(x)^i$.

(2) Show that (formally)

$$\zeta_{p,i}(s) = \mu_{\eta,i}(\exp(ps)). \tag{1.7.1}$$

(3) From (2), deduce that $\zeta_{p,i}(s)$ extends to a p-adic analytic function for $s \in p^{-\frac{p-2}{p-1}}\mathfrak{m}_{\mathbb{C}_p}$.

Remark 5. One sees from this exercise that the classical p-adic L-function only captures part of the information provided. Even knowing the convergence of $\zeta_{p,i}(s)$ for $s \in p^{-\frac{p-2}{p-1}}\mathfrak{m}_{\mathbb{C}_p}$, it is far from enough to deduce the integrality of $\mu_\eta^{\{p\}}$. For more discussion in this direction, see the post

`https://mathoverflow.net/questions/435265/why-p-adic-measures`.

Exercise 1.8 (Volume of ideles class group versus residue of Dedekind zeta values)**.** Let F be a number field with r_1 real embeddings and r_2 pairs of complex embeddings. Let $\mathbb{A}_F^\times$ be the group of ideles and $\mathbb{A}_F^{\times,1} := \{x \in \mathbb{A}_F^\times \mid |x| = 1\}$ be the subgroup of norm one elements. We have stated (and proved in the quadratic case) of the analytic class number formula, for the Dedekind zeta function $\zeta_F(s)$ at $s = 1$:

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} \cdot h_F \mathrm{Reg}_F}{w_F\sqrt{|\Delta_F|}}, \tag{1.8.1}$$

where h_F is the class number, Reg_F is the regulator for units of F, w_F is the number of roots of unity contained in F, and Δ_F is the discriminant of F.

(1) Using the functional equation of Dedekind zeta function to deduce from (1.8.1) the following analytic class number formula at $s = 0$:

$$\lim_{s\to 0} s^{-r_1-r_2+1}\zeta_F(s) = -\frac{h_F \cdot \mathrm{Reg}_F}{w_F}.$$

(2) Show that the right hand side of (1.8.1) can be interpreted as $\mathrm{Vol}(\mathbb{A}_F^{\times,1}/F^\times)$, if we provide the Haar measure on $\mathbb{A}_F^{\times,1}$ so that under the product decomposition $\mathbb{A}_F^\times = \mathbb{A}_F^{\times,1} \times \mathbb{R}^\times$ (where $\mathbb{R}^\times$ is provided with the measure $\frac{dx}{x}$) admits the following Haar measure:

- at a real place v of F, the Haar measure on $F_v^\times$ is $\frac{dx}{|x|}$,
- at a complex place v of F, the Haar measure on $F_v^\times \simeq \mathbb{C}^\times$ is $\frac{2dx\wedge dy}{|x^2+y^2|} = \frac{2drd\theta}{r}$,
- at a p-adic place v of F with different ideal $\mathfrak{d}_v \subseteq F_v$, the Haar measure on $F_v^\times$ is so that volume of $\mathcal{O}_{F_v}^\times$ is $||\mathfrak{d}_v||^{-\frac{1}{2}}$, where $||\mathfrak{d}_v|| = \#(\mathcal{O}_{F_v}/\mathfrak{d}_v)$.