Special values of L-functions 1

—— Special values of Dirichlet L-functions

Style of this course:

Conjectures: Observation/philosophy $\to$ examples $\to$ conjectures $\to$ key examples/compatibility checks

Proofs: Classical proofs/proofs in examples $\to$ formal framework $\to$ fancy proof $\to$ back to example "reality checks"

§1. Dirichlet L-functions and their special values.

Definition Riemann zeta function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$ (Re $s>1$)

(Euler) $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\dots$, $\zeta(2n) \in \mathbb{Q}^\times \cdot \pi^{2n}$

(Apéry 1978) $\zeta(3)$ is irrational.

Conjecture: $\pi, \zeta(3), \zeta(5), \zeta(7), \cdots$ are algebraically independent,

i.e. if $P(x, y_3, y_5, \cdots) \in \mathbb{Q}[x, y_3, y_5, \cdots]$ is a polynomial s.t.

$P(\pi, \zeta(3), \zeta(5), \zeta(7), \cdots) = 0$, then $P \equiv 0$.

(functional equations and Riemann hypothesis will be discussed below.)

Definition Fix $N \in \mathbb{Z}_{>0}$, a character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is called a Dirichlet character of conductor N

It is called primitive if it does not factors through $(\mathbb{Z}/M\mathbb{Z})^\times$ for any $M|N$.

For a Dirichlet character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, define

$$L(\eta, s) = \sum_{(n,N)=1} \frac{\eta(n)}{n^s} = \prod_{\substack{p \nmid N \\ \text{prime}}} \frac{1}{1-\eta(p)p^{-s}} \qquad \text{when } \mathrm{Re}(s) > 1$$

Question: Special values of $L(\eta, s)$?

E.g. $\eta : (\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $\eta(-1) = -1$.

$$L(\eta, 1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \arctan 1 = \frac{\pi}{4}.$$

E.g. $\eta:(\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$, $\eta(3)=\eta(5)=-1$, $\eta(-1)=\eta(3)\eta(5)=1$.

$$L(\eta,1) = 1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\frac{1}{15}+\cdots$$

Naïve approach: Consider $f(x) = x-\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{7}x^7+\cdots$

then $f'(x) = 1-x^2-x^4+x^6+\cdots = \dfrac{1-x^2-x^4+x^6}{1-x^8}$

$$\Rightarrow f(x) = \int \frac{1-x^2-x^4+x^6}{1-x^8}dx = \frac{\sqrt{2}}{4}\left(\ln\left|x^2+\sqrt{2}x+1\right| - \ln\left|x^2-\sqrt{2}x+1\right|\right)$$

$$\Rightarrow L(\eta,1) = f(1) = \frac{\sqrt{2}}{4}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)$$

Rmk: $\sqrt{2}+1$ is the fundamental unit in $\mathbb{Z}[\sqrt{2}]$; $2\sqrt{2}$ is $\sqrt{d_{\mathbb{Q}(\sqrt{2})}}$

- Say η is even if $\eta(-1)=1$, and η is odd if $\eta(-1)=-1$.

Then: When η is even, for $m\in\mathbb{Z}_{>0}$, $L(\eta,2m)\in\overline{\mathbb{Q}}^\times\cdot\pi^{2m}$

When η is odd, for $m\in\mathbb{Z}_{>0}$, $L(\eta,2m-1)\in\overline{\mathbb{Q}}^\times\cdot\pi^{2m-1}$

"period type" (Deligne's conjecture)

* When $\eta:(\mathbb{Z}/N\mathbb{Z})^\times\to\{\pm1\}$ is primitive quadratic character s.t. $\eta(-1)=1$

(then $\eta:(\mathbb{Z}/N\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})\to\{\pm1\}$ corresponds to a real quad. field F.)

$L(\eta,1)\in\mathbb{Q}^\times\cdot\sqrt{d_F}\cdot\ln|u_F|$, where $u_F\in\mathcal{O}_F^\times$ is a fundamental unit.

"regulator type" (Beilinson's conjecture)

Theme of this course: Understand the philosophy behind these conjectures and possible generalizations of them.

§2. Functional equations of Dirichlet L-functions

- Let $\eta:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$ be a primitive character of conductor N

$$L(\eta,s) = \prod_{\substack{p\nmid N\\ p\text{ prime}}} \frac{1}{1-\eta(p)p^{-s}} \quad \leftarrow L_p(\eta,s)$$

For the purpose of functional equations, we put

$$L_\infty(\eta,s) = \begin{cases}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } \eta(-1)=1 \quad (\text{Put } \delta=0)\end{cases}$$

$$L_\infty(\eta,s) = \begin{cases} \\ \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right) & \text{if } \eta(-1)=-1 \quad (\text{Put } \delta=1)\end{cases}$$

Remark: In the "standard theory", we put $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$

satisfying $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1)$ $\quad (*)$

Put $\Lambda(\eta,s) := L(\eta,s)\cdot L_\infty(\eta,s)$

Theorem Every Dirichlet L-function $L(\eta,s)$ admits a $\begin{cases}\text{holomorphic extension (if } \chi\neq 1) \\ \text{meromorphic extension (if } \chi=\mathbb{1})\end{cases}$ to $s\in\mathbb{C}$

and a function equation

$$\Lambda(\eta,s) = \varepsilon(\eta,s)\cdot\Lambda(\eta,1-s)$$

where $\varepsilon(\eta,s) = \underbrace{G(\eta)}_{\text{Gauss sum}}\cdot N^{-s}/i^{\delta}$ $\qquad$ (Important: $|\varepsilon(\eta,\tfrac{1}{2})|=1$)

Gauss sum $|G(\eta)| = \sqrt{N}$.

Goal today: Assuming functional equation $\Rightarrow L(\eta,m)\in\overline{\mathbb{Q}}^\times\cdot\pi^m$ if $\begin{matrix} m \text{ even and } \eta(-1)=1 \\ m \text{ odd and } \eta(-1)=-1\end{matrix}$

§3 Techniques for special values

Define $\Gamma(s) = \displaystyle\int_0^{+\infty} e^{-t}t^s\cdot\frac{dt}{t}$ if $\mathrm{Re}(s)>0$

Then (1) $\Gamma(n) = (n-1)!$ if $n\geq 1$

(2) $\Gamma(s+1) = s\cdot\Gamma(s)$ $\forall s$ $\Rightarrow$ meromorphic continuation of $\Gamma(s)$ with a simple pole at $s\in\mathbb{Z}_{\leq 0}$

(3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\Rightarrow$ $\Gamma\left(m+\frac{1}{2}\right) = \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\frac{1}{2}\cdot\sqrt{\pi}$. for $m\in\mathbb{Z}_{\geq 1}$

$\Gamma\left(-m+\frac{1}{2}\right) = \left(-m+\frac{1}{2}\right)^{-1}\left(-m+\frac{3}{2}\right)^{-1}\cdots\left(-\frac{1}{2}\right)^{-1}\cdot\sqrt{\pi}$ for $m\in\mathbb{Z}_{\geq 0}$

• We have $\displaystyle\int_0^{+\infty} e^{-nt}t^s\cdot\frac{dt}{t} = n^{-s}\cdot\Gamma(s)$

For a Dirichlet character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $\quad$ (put $\eta(n)=0$ if $(n,N)\neq 1$)

$$\Gamma(s)L(\eta,s)=\Gamma(s)\sum_{\substack{(n,N)=1\\ n\geq 1}}\frac{\eta(n)}{n^s}=\int_0^{+\infty}\sum_{\substack{(n,N)=1\\ n\geq 1}}\eta(n)e^{-nt}\,t^s\frac{dt}{t}=\int_0^{+\infty}\frac{\sum_{n=0}^{N-1}\eta(n)e^{-nt}}{1-e^{-Nt}}\cdot t^s\frac{dt}{t}$$

Key technical lemma For a smooth function $f(t)\in C^\infty(\mathbb{R}_{\geq 0})$ that is rapidly decreasing as $t\to+\infty$

(learned from Colmez's Tsinghua notes)

i.e. $t^n\partial_t^m(f)(t)\to 0$ as $t\to+\infty$ for any $m,n\in\mathbb{Z}_{\geq 0}$.

the function
$$L(f,s):=\frac{1}{\Gamma(s)}\int_0^{+\infty}f(t)t^s\frac{dt}{t}\qquad(\text{when } \mathrm{Re}(s)>0)$$

has an analytic continuation to $\mathbb{C}$, and $L(f,-n)=(-1)^n f^{(n)}(0)$ for any $n\in\mathbb{Z}_{\geq 0}$

Rmk: For Dirichlet functions, the "correct" L-factor at ∞ is $\Gamma_{\mathbb{R}}(s)$ or $\Gamma_{\mathbb{R}}(s+1)$, but to access the special values & for rationality, it seems that we need $\Gamma(s)L(f,s)$?!

Proof: Choose $\phi\in C^\infty([0,+\infty))$ s.t. $\phi(t)=1$ if $t\in[0,1]$, $\phi(t)=0$ if $t\geq 2$.

Write $f=f_1+f_2$ for $f_1=\phi f$ and $f_2=(1-\phi)\cdot f$

Then $\int_0^{+\infty}f_2(t)t^s\frac{dt}{t}=\int_1^{+\infty}f_2(t)t^s\frac{dt}{t}$ is holomorphic for $s\in\mathbb{C}$

$$\Rightarrow L(f_2,-n)=\left(\frac{1}{\Gamma(s)}\int_1^{+\infty}f_2(t)t^s\frac{dt}{t}\right)\Bigg|_{s=-n}=0\qquad(\text{as } \Gamma(s) \text{ has a pole at } s=-n)$$

$$f_2^{(n)}(0)=0.\qquad\checkmark$$

For f_1, we consider (when $\mathrm{Re}(s)>0$)

$$L(f_1,s)=\underbrace{\frac{1}{\Gamma(s)}\left(f_1(t)\frac{t^s}{s}\right)\Bigg|_0^{+\infty}}_{=0}-\frac{1}{s\Gamma(s)}\int_0^{+\infty}f_1'(t)t^{s+1}\frac{dt}{t}$$

$$=-\frac{1}{\Gamma(s+1)}\int_0^{+\infty}f_1'(t)t^{s+1}\frac{dt}{t}=-L(f_1',s+1)$$

By induction, $L(f_1,-n)=(-1)^{n+1}\cdot L(f_1^{(n+1)},1)=(-1)^{n+1}\cdot\int_0^{+\infty}f_1^{(n+1)}(t)dt$

$$=(-1)^{n+1}\left(f_1^{(n)}(t)\right)\Big|_0^{+\infty}=(-1)^n\cdot f_1^{(n)}(0).\qquad\square$$

§4 Special values of Dirichlet L-functions, and their algebraicity

Corollary: For a Dirichlet character η, primitive of conductor N,

$$L(\eta,-n)\in\mathbb{Q}(\eta) \quad \text{for } n\in\mathbb{Z}_{\geq 0}$$

Further discussion: Write $\eta(-1)=(-1)^{\delta}$ for $\delta\in\{0,1\}$

$$f_\eta(t)=\frac{\sum_{n=1}^{N-1}\eta(n)\cdot e^{-nt}}{1-e^{-Nt}}$$

$$\Rightarrow f_\eta(-t)=\frac{\sum_{n=1}^{N-1}\eta(n)e^{nt}}{1-e^{Nt}}=\frac{e^{Nt}\cdot\sum_{n=1}^{N-1}\eta(n-N)\cdot e^{(n-N)t}}{e^{Nt}\cdot\left(e^{-Nt}-1\right)}\underset{n'=N-n}{=}\frac{\eta(-1)\sum_{n'=1}^{N-1}\eta(n')e^{-n't}}{-\left(1-e^{-Nt}\right)}$$

$$=-\eta(-1)\cdot f_\eta(t)$$

So when $\eta(-1)=1$, f_η is an odd function $\Rightarrow L(\eta,-n)=(-1)^n f_\eta^{(n)}(0)=0$ if n even

$\eta(-1)=-1$ —— even function $\Rightarrow L(\eta,-n)=$ —— $=0$ if n odd.

This is expected: comparing with the functional equation

$$\pi^{-\frac{s+\delta}{2}}\Gamma\left(\tfrac{s+\delta}{2}\right)\cdot L(\eta,s)=\varepsilon(\eta,s)\cdot L(\eta^{-1},1-s)\cdot\pi^{-\frac{1-s+\delta}{2}}\Gamma\left(\tfrac{1-s+\delta}{2}\right)$$

$$\Rightarrow \quad \underbrace{\Gamma\left(\tfrac{s+\delta}{2}\right)}_{\text{pole if } n\equiv\delta \bmod 2}\cdot L(\eta,s)=\frac{G(\eta)\cdot N^{-s}}{i^{\delta}}\cdot\underbrace{L(\eta^{-1},1-s)}_{\neq 0 \text{ unless } \eta=\mathbb{1},\, s=0}\cdot\pi^{s-\frac12}\underbrace{\Gamma\left(\tfrac{1-s+\delta}{2}\right)}_{\text{no poles/zeros}} \quad (*)$$

Take $s=-n$ for $n\in\mathbb{Z}_{\geq 0}$

So should expect $L(\eta,-n)=0$ if $n\equiv\delta \bmod 2$. (and also $\zeta(0)\in\mathbb{Q}^{\times}$)

When $s=n\in\mathbb{Z}_{\geq 1}$ and $s\equiv\delta \pmod 2$ $\rightsquigarrow$ get $\mathbb{Q}^{\times}\cdot L(\eta,n)\in\overline{\mathbb{Q}}^{\times}\cdot\mathbb{Q}(\eta)\cdot\pi^{n-\frac12}\cdot\mathbb{Q}^{\times}\sqrt{\pi}$

$$\Rightarrow L(\eta,n)\in\overline{\mathbb{Q}}\cdot\pi^{n}$$

Finer properties: $\eta:(\mathbb{Z}/N\mathbb{Z})^{\times}\longrightarrow\mathbb{Q}^{alg,\times}\subseteq\mathbb{C}^{\times}$ ($\mathbb{Q}^{alg}$ = algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$)

then $Gal(\mathbb{Q}^{alg}/\mathbb{Q})$ acts on η as $\eta\rightsquigarrow\sigma\circ\eta$

Corollary (1) For $\sigma\in Gal(\mathbb{Q}^{alg}/\mathbb{Q})$, $L(\sigma\circ\eta,-n)=\sigma\left(L(\eta,-n)\right)$ when $n\in\mathbb{Z}_{\geq 0}$.

(2) When $n \in \mathbb{Z}_{\geq 1}$, $n \equiv \delta \bmod 2$, if $\sigma \in \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}(\zeta_N))$

$$\frac{L(\sigma\circ\eta, n)}{(2\pi i)^n} = \sigma\left(\frac{L(\eta, n)}{(2\pi i)^n}\right) \qquad (\star)$$

Proof: (1) is clear. For (2), using $(*)$ when $s=n\in\mathbb{Z}_{\geq 1}$ and $n \equiv \delta \pmod 2$)

$$L(\eta, n) \in \frac{G(\eta)}{i^\delta} \cdot L(\eta^{-1}, 1-n)\cdot \pi^n \cdot \mathbb{Q}^\times$$

Or equivalently, $\dfrac{L(\eta,n)}{(2\pi i)^n} \in G(\eta)\cdot L(\eta^{-1}, 1-n)\cdot \mathbb{Q}^\times$ as $n \equiv \delta \pmod 2$)

Need to show $\quad G(\sigma\circ\eta) = \sigma(G(\eta))$. This is clear.

It seems clear that for general $\sigma \in \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$, $(\star)$ does not hold.

I don't have a good understanding of this now, which does appear in literature from time to time

§5 Kummer congruence and Kubota-Leopoldt p-adic L-functions

Story: determine L-values, up to $\bar{\mathbb{Q}}^\times$, then up to $\mathbb{Q}^\times$, and finally understand congruences of L-values

· Recall: $f_{\mathbb{1}}(t) = \dfrac{1}{1-e^{-t}} = \dfrac{1}{t}\displaystyle\sum_{n=0}^{+\infty} B_n \frac{t^n}{n!}$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30}, \cdots \quad B_{2m+1} = 0 \ (\text{when } m \geq 1)$$

$$\zeta(1-n) = -f_{\mathbb{1}}^{(n-1)}(0) = -\frac{B_n}{n}$$

Kummer congruence. Let $k \in \mathbb{Z}_{\geq 1}$ and let integers $n_1, n_2 \geq k$ such that $n_1 \equiv n_2 \bmod (p-1)p^{k-1}$, and that $p-1 \nmid n_1$ ($\Leftrightarrow p-1 \nmid n_2$),

then $\dfrac{B_{n_1}}{n_1} \equiv \dfrac{B_{n_2}}{n_2} \bmod p^k$; or equivalently $\zeta(1-n_1) \equiv \zeta(1-n_2) \bmod p^k$.

Will prove this in the next lecture

Later: $v_p(\zeta(1-n))$ is related to ideal class groups.