Special values of L-functions 11

—— Introduction to motives II.

§1 Motives

Definition An *effective motive* is a pair (X,e),

where $X \in \mathrm{SmProj}/k$ and $e \in \mathrm{Corr}^0(X,X)$ is an idempotent, i.e. $e^2=e$.

$\mathrm{Mot}^{\mathrm{eff}}(k)$ = category of effective motives

$$\mathrm{Mor}_{\mathrm{Mot}^{\mathrm{eff}}(k)}\big((X,e),(Y,f)\big) := f \circ \mathrm{Corr}^0(X,Y) \circ e.$$

means direct sum of all cohom.

Realizations:

$$\begin{tikzcd}[column sep=large]
 & \mathrm{Fil}\text{-}\mathrm{Vect}/k \\
\mathrm{Mot}^{\mathrm{eff}}(k) \arrow[ur, "\mathrm{deRham}"] \arrow[r, "\mathrm{Betti}"] \arrow[dr, "\text{étale}"'] & \mathbb{Z}\text{-}\mathrm{Mod} \ (\text{if } k=\mathbb{C}) \\
 & \mathbb{Z}_\ell[[\mathrm{Gal}_k]]\text{-}\mathrm{Mod} \ (\text{if } \ell \neq \mathrm{char}\, k)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M_{dR} = eH^*_{dR}(X/k) \\
M=(X,e) \arrow[ur, mapsto] \arrow[r, mapsto] \arrow[dr, mapsto] & M_B = eH^*_B(X(\mathbb{C})^{an},\mathbb{Z}) \\
 & M_{\text{ét},\ell} = eH^*_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)
\end{tikzcd}$$

If we fix the degree n, then we have all the properties of $H^n(X)$ for all effective motives

Examples ① $\mathbb{1} = (\mathrm{Spec}\, k, \mathrm{id})$.

② Let k'/k be a Galois extension with $G = \mathrm{Gal}(k'/k)$

$$\mathbb{Q}[G] = \bigoplus V_i \quad \text{for each } V_i \text{ irred. (left } G\text{-action)}$$

Then $\exists\, e_i \in \mathbb{Q}[G]$ idempotent s.t. $V_i = \mathbb{Q}[G]\cdot e_i$

$$M(V_i) := (\mathrm{Spec}\, k', e_i) \rightsquigarrow M_{et,\ell}(V_i) = V_i \otimes \mathbb{Q}_\ell$$

③ Direct sum: $(X,p) \oplus (Y,q) = (X \sqcup Y, p \oplus q)$.

Tensor product: $(X,p) \otimes (Y,q) = (X \times Y, p \otimes q)$

④ $X \in \mathrm{SmProj}/k$ pure of dim d, and $e \in X(k) \neq \emptyset$.

Define $p_{0,X} := e \times X$ and $p_{2d,X} := X \times e$

Their actions on cohomology: (e.g. $X/\mathbb{C}$)

$$\begin{tikzcd} p_{0,X*} : H^i_B(X(\mathbb{C})^{an},\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\text{factors as}"'] & H^i_B(X(\mathbb{C})^{an},\mathbb{Q}) \\ H^i_B(\{e\},\mathbb{Q}) \arrow[r, "p_1^*"] & H^i_B(\{e\}\times X,\mathbb{Q}) \arrow[u, "\simeq\ p_{2!}"'] \end{tikzcd} \qquad \begin{tikzcd} & e\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

only $i=0$ survived.

Similarly, $p_{2d,X}$ behaves like projection to $H^{2d}_B(X(\mathbb{C})^{an},\mathbb{Q})$

<u>Fact</u>: $p_{0,X}$ and $p_{2d,X}$ are "orthogonal idempotents", i.e. $p_{0,X}^2 = p_{0,X}$, $p_{2d,X}^2 = p_{2d,X}$, $p_{0,X}p_{2d,X} = p_{2d,X}p_{0,X}$

$(X, p_{0,X}) \simeq \mathbb{1}$

For $X = \mathbb{P}^1$, $(\mathbb{P}^1, [\mathbb{P}^1\times\{0\}]) =: \mathbb{Z}(-1) \approx H^2(\mathbb{P}^1)$

<u>Fact</u>: For any X above, $(X, p_{2d,X}) \simeq \mathbb{Z}(-d) = \mathbb{Z}(-1)^{\otimes d}$

<u>Conjecture</u> (Künneth decomposition) $X \in \mathrm{SmProj}/k$ irreducible of dim d

$\exists$ commuting idempotents $e_0, \cdots, e_{2d} \in \mathrm{Corr}^0(X,X)_{\mathbb{Q}}$

s.t. "(X, e_i) corresponds to $H^i(X)$" in the sense that $\forall \ell$, $(X,e_i)_{et} = H^i_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)$.

<u>Remark</u>: (1) True for X curve with $P \in X(k)$,

$$e_1 = id - e_0 - e_2 = [\Delta_X] - [P\times X] - [X\times P] \in CH^1(X\times X).$$

(2) True for X abelian variety, this follows from Lieberman's trick:

mult_ℓ acts on $H^n(X)$ by mult. by ℓ^n. (Also true for surfaces.)

(3) Okay for X surface b/c $H^1(X)$ and $H^3(X)$ are essentially $\mathrm{Pic}^0(X)$.

* <u>Existence of Correspondence vs. Tate conjecture</u>

Suppose $X, Y \in \mathrm{SmProj}/\mathbb{Q}$ $\rightsquigarrow$ $H^m_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$, $H^n_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$
$\circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$ $\quad$ $\circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$

In some cases, $\exists$ Galois direct summands $M \subseteq H^m_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$, $N \subseteq H^n_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$

s.t. $M \simeq N$ as irreducible Galois representations (and assume that the direct sum complement has no factors isomorphic to M and N.)

Consider Frobenius eigenvalues and weights $\Rightarrow$ $m = n$.

Then Tate conjecture $\Rightarrow \exists$ correspondence $T \in \mathrm{Corr}^0(X,Y)$

$$[T]_* : H^m_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \twoheadrightarrow M \overset{\text{some isom.}}{\xrightarrow{\sim}} N \subseteq H^m_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$$

(Proof: Say $d_X := \dim X$, $d_Y := \dim Y$.

Condition $\Rightarrow \mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}}}\left(H^m_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell), H^m_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)\right) \neq 0$

$$= \Big(\underbrace{H^m_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^*}_{\substack{\| \text{ duality} \\ H^{2d_X-m}_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d_X))}} \otimes H^m_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \Big)^{\mathrm{Gal}_{\mathbb{Q}}} \ni \xi$$

By Künneth formula, $H^{2d_X}_{ét}((X\times Y)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d_X)) = \bigoplus_{i=0}^{2d_X} H^i_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(d_X)) \otimes H^{2d_X-i}_{ét}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$

ξ in the $\mathrm{Gal}_{\mathbb{Q}}$-invariant part with $i = 2d_X - m$

By Tate conjecture for $X \times Y \Rightarrow \exists\, \xi^{alg} \in CH^m(X\times Y) = \mathrm{Corr}^0(X,Y)$

s.t. $[\xi^{alg}]_*$ is a multiple of ξ. $\square$)

<u>Special case</u>: If ρ is an irred. Galois rep'n, $\rightsquigarrow H^n_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)[\rho] = \rho$-isotypical component.

Tate conj $\Rightarrow \exists$ a projector $pr_\rho \in \mathrm{Corr}^0(X,X)$ s.t. $pr_\rho H^n_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = H^n_{ét}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)[\rho]$

<u>Definition</u>. $\mathrm{Mot}_{/k} := \mathrm{Mot}^{\mathrm{eff}}(k)$ "formally inverting $\mathbb{Z}(-1)$"

<u>Object</u>: (X, e, n) avatar of $eH^{\cdot}(X)(n)$

<u>Morphism</u> $\mathrm{Mor}_{\mathrm{Mot}_{/k}}\left((X,e,n),(X',e',n')\right) := \varinjlim_{m\to\infty} \mathrm{Mor}\left((X,e)\otimes\mathbb{Z}(n-m), (X',e')\otimes\mathbb{Z}(n'-m)\right)$

Similarly, we have

$$\begin{array}{lll} & \xrightarrow{\text{de Rham}} \mathrm{Fil}\text{-}\mathrm{Vect}_{/k} & \\ \mathrm{Mot}_{/k} & \xrightarrow{\text{étale}} \mathbb{Z}_\ell\text{-}\mathrm{Mod} & \ell \neq \mathrm{char}\, p \\ & \xrightarrow{\text{Betti}} \mathbb{Z}\text{-}\mathrm{Mod} & k = \mathbb{C} \end{array}$$

+ comparison.

<u>Definition</u>. There's a natural duality $M \longmapsto \check{M} : \mathrm{Mot}_{/k} \longrightarrow \mathrm{Mot}_{/k}$

For $M = (X, e, n)$ with X irreducible of dim d,

define $\check{M} := (X, {}^t e, d-n)$ to be its dual.

Then for all realizations $? \in \{dR, ét, B\}$, $(\check{M})_? = (M_?)^\vee$.

<u>Example</u>: Realizations of $M = \mathbb{Z}(-1)$. weight 2

$M_B = H^2(\mathbb{CP}^1, \mathbb{Z}) = \frac{1}{2\pi i}\mathbb{Z}$, it is purely imaginary b/c F_∞ acts on it by -1.

$M_{dR} = H^2(\mathbb{P}^1/\mathbb{Q}) = \mathbb{Q}$

$M_{ét} = H^2_{ét}(\mathbb{P}^1_{\mathbb{Q}^{alg}}, \mathbb{Z}_\ell) = (\varprojlim_n \mu_{\ell^n}(\mathbb{C}))^\vee \simeq \mathbb{Z}_\ell(-1)$. with standard basis $(e^{2\pi i/\ell^n})_{n\geq 0}$ depending on the embedding $\mathbb{Q}^{alg} \subseteq \mathbb{C}$.

<u>The comparisons</u>: $M_B \otimes \mathbb{C} \simeq M_{dR} \otimes \mathbb{C}$

$$\frac{1}{2\pi i}\cdot 2\pi i = 1 \longleftrightarrow 1$$

$$M_B^* \otimes \mathbb{Z}_\ell \simeq M^*_{et,\ell}$$

$$2\pi i \longmapsto (e^{2\pi i/\ell^n})_{n\geq 0}$$

For $M \in Mot/\mathbb{Q}$, assume that M is pure of weight $w = i - 2m$ (i.e. a quotient of $H^i(X)(m)$)

$\rightsquigarrow M_{ét,\ell}$ is a $\mathbb{Z}_\ell$-mod, equipped with $Gal_{\mathbb{Q}}$-action

$\rightsquigarrow L(M,s) := \prod_{p \text{ prime}} L_p(M_{et,\ell}, s)$

<u>Question</u>: What about factors at ∞?

<u>Slogan</u>: Hodge structure is the ∞-analogue of Galois representations.

§2 <u>Hodge structures</u>

<u>Basic model</u>: $H^n_B(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes \mathbb{C} \cong H^n_{dR}(X/\mathbb{Q}) \otimes \mathbb{C}$

$$\cong \bigoplus_{p+q=n} H^{pq}.$$

<u>Definition</u>. For $A \subseteq \mathbb{R}$ a subring (typically $\mathbb{Z}$ or $\mathbb{Q}$), a <u>Hodge structure over A</u> consists of ① A locally free A-module V

② A decomposition $V_{\mathbb{C}} := V \otimes_A \mathbb{C} \simeq \bigoplus_{p,q} V^{pq}$ with $\overline{V^{pq}} = V^{qp}$

We say V has pure weight n if $V^{pq} \neq 0 \Rightarrow p+q = n$

In this case, ② is equivalent to ②′ giving a descending filtration $F^i V_{\mathbb{C}}$ s.t. $F^p V_{\mathbb{C}} \cap \overline{F^{n+1-p} V_{\mathbb{C}}} = 0$

② ⇒ ②′, put $F^p V_{\mathbb{C}} := \bigoplus_{p' \geq p} V^{p', n-p'}$

②′ ⇒ ② put $V^{pq} = F^p V_{\mathbb{C}} \cap \overline{F^{n-p} V_{\mathbb{C}}}$

<u>Alternative definition</u>: Recall the following equivalence of categories

$\{$f.dim'l graded $\mathbb{C}$-vector spaces$\}$ $\longleftrightarrow$ $\{$Representations$/\mathbb{C}$ of $\mathbb{G}_m\}$

$V = \bigoplus V^p$ $\qquad$ $\mathbb{G}_m$ acts on V^p by $x \mapsto \text{mult}_{x^{-p}}$ will explain the sign convention later.

$\{$f.dim'l bigraded $\mathbb{C}$-vector spaces$\}$ $\longleftrightarrow$ $\{$Representations $/\mathbb{C}$ of $\mathbb{G}_m \times \mathbb{G}_m\}$

$V = \bigoplus V^{pq}$ $\qquad$ $(z,w) \in (\mathbb{G}_m \times \mathbb{G}_m)(\mathbb{C})$ acts on V^{pq} via $z^{-p} w^{-q}$

$\iff \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \longrightarrow GL_{\mathbb{C}}(V)$ (*)

$$V^{pq} = \overline{V^{qp}} \longleftrightarrow \begin{array}{ccc} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \times V_{\mathbb{C}} & \xrightarrow{\text{act}} & V_{\mathbb{C}} \\ \downarrow & & \downarrow \text{conj} \\ \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \times V_{\mathbb{C}} & \xrightarrow{\text{act}} & V_{\mathbb{C}} \end{array} \qquad \begin{array}{ccc} \underset{v \in V^{pq}}{(z,w,v)} & \longmapsto & z^{-p} w^{-q} v \\ \downarrow & & \downarrow \\ \underset{\bar v \in V^{qp}}{(\bar w, \bar z, \bar v)} & \longmapsto & \bar z^{-p} \bar w^{-q} \bar v \end{array}$$

So $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \longrightarrow GL(V)_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$

$\circlearrowleft$ $(z,w) \mapsto (\bar w, \bar z)$ $\qquad$ $\circlearrowleft$ complex conj

We can then descent (*) to a homomorphism $h: \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow GL_{\mathbb{R}}(V)$

So we have an equivalence: $\{\mathbb{R}$-Hodge structures$\}$ $\longrightarrow$ $\{\mathbb{S}$-rep'n$/\mathbb{R}\}$.

<u>Remark</u>: We may combine M_B, M_{dR} together to get $\text{Mot}_{/\mathbb{R}} \longrightarrow \{\mathbb{S}\text{-rep'ns}/\mathbb{R}\}$

This looks more like $M_{et,\ell}: \text{Mot}_{/\mathbb{Q}} \longrightarrow \{\mathbb{Z}_\ell[[\text{Gal}_{\mathbb{Q}}]]\text{-mods}\}$.

<u>Mumford-Tate conjecture</u>:

Recall $M \in \text{Mot}^{\text{pure}}_{/\mathbb{Q}}$ $\xrightarrow{r_{et,\ell}}$ $\text{Rep}_{\mathbb{Q}_\ell}(\text{Gal}_{\mathbb{Q}})$; $\xrightarrow{r_{\text{Hodge}}}$ $\{$pure Hodge structure$/\mathbb{Q}\}$ — how to compare?

Definition. If V is a $\mathbb{Q}$-Hodge structure of pure weight n, consider

$$h_V : \mathbb{S} \longrightarrow GL(V) \times_{\mathbb{Q}} \mathbb{R}$$

Define the Mumford-Tate group $MT(V)$ to be the minimal algebraic $\mathbb{Q}$-subgroup of $GL(V)$ whose base change to $\mathbb{R}$ contains $\mathrm{Im}(h_V)$

Example: E elliptic curve $/\mathbb{Q}$ $\rightsquigarrow V = H^1(E,\mathbb{Q})$

* If $\mathrm{End}_{\overline{\mathbb{Q}}}(E) = \mathbb{Z}$, i.e. E has no CM then $MT(V) = GL(V) = GL_{2,\mathbb{Q}}$
* If $\mathrm{End}_{\overline{\mathbb{Q}}}(E) = \mathcal{O}$, an order in $\mathbb{Q}(\sqrt{-D})$, $\mathcal{O} \hookrightarrow H^1(E,\mathbb{Q}) = V$

then $MT(V) = GL(V)^{\mathcal{O}} = \{ g \in GL(V) \mid \forall h \in \mathcal{O} \text{ s.t. } h \circ g = g \circ h \}$

$$= \mathrm{Res}_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}} \mathbb{G}_m .$$

For $M \in \mathrm{Mot}/\mathbb{Q} \rightsquigarrow \rho_\ell : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL(M_{et,\ell} \otimes \mathbb{Q}_\ell)$

$$\mathfrak{g}_{M,et,\ell} := \mathrm{Lie}\left(\overline{\mathrm{Im}\,\rho_\ell}^{\text{Zariski closure}}\right)$$

Mumford-Tate conjecture(?) Under the Betti-étale comparison isomorphism

$$M_B \otimes \mathbb{Q}_\ell \cong M_{et,\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$$\Longrightarrow \quad GL(M_B) \times \mathbb{Q}_\ell \cong GL(M_{et,\ell}[\tfrac{1}{\ell}])$$

$$\begin{array}{ccc} \cup\!\!| & & \cup\!\!| \\ MT(M_{Hdg}) \times \mathbb{Q}_\ell & = & \mathfrak{g}^{\circ}_{M,et,\ell} \end{array}$$

Known cases (Deligne) When $M = H^1(A,\mathbb{Q})$ for A abelian variety,

$$\mathfrak{g}_{M,et,\ell} \subseteq \mathrm{Lie}(MT(V)^{\circ}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$