Special values of L-functions 14

——— Algebraic Hecke character versus p-adic ones

<u>Goal</u>: Find a theory that generalizes

* Dirichlet characters
* characters of ideal class group
* cyclotomic characters

§1. Hecke characters

<u>Definition</u> Let F be a number field. A Hecke character is a continuous character

$$\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times.$$

It is the same as giving for each place $v \in M_F$ a continuous character $\chi_v : F_v^\times \to \mathbb{C}^\times$

s.t. ① for all but finitely many v, χ_v is unramified,

i.e. it factors as $\chi_v : F_v^\times \twoheadrightarrow \mathcal{O}_{F_v}^\times / \mathcal{O}_{F_v}^\times \longrightarrow \mathbb{C}^\times$

② for every $a \in F^\times$, $\prod_{v \in M_F} \chi_v(a) = 1$ (it is a finite product)

(ocassionally for a complex embedding $\sigma : F \hookrightarrow \mathbb{C}$, better to have both χ_σ and $\chi_{c\sigma}$ s.t. $\chi_{c\sigma}(a) = \chi_\sigma(c(a))$.)

For each Hecke character χ, define its L-function to be

* for v a finite place: $L_v(\chi_v, s) = \begin{cases} \dfrac{1}{1 - \chi_v(\varpi_v) N v^{-s}} & \text{if } \chi_v \text{ is unramified} \\ 1 & \text{if } \chi_v \text{ is ramified.} \end{cases}$

* for $v = \mathbb{R}$. $\chi_v : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$ wt normalization ⇝ wt $= n_\sigma$

$\sigma : F_v \xrightarrow{\sim} \mathbb{R}$, $\chi_v(a) = |a|^{-n_\sigma}$ or $\chi_v(a) = \mathrm{sgn}(a)|a|^{-n_\sigma}$ but allowing $n_\sigma \in \mathbb{R}$

$$L_v(\chi_v, s) = \begin{cases} \Gamma_{\mathbb{R}}(s - n_\sigma) & \chi_v = |\cdot|^{-n_\sigma} \\ \Gamma_{\mathbb{R}}(s - n_\sigma + 1) & \chi_v = \mathrm{sgn}(\cdot)|\cdot|^{-n_\sigma} \end{cases}$$

· for $v = \mathbb{C}$. $\chi_v : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$

$\sigma : F_v \longrightarrow \mathbb{C}$ $\quad \chi_v(re^{i\theta}) = r^{2l_v} \cdot e^{i m_v \theta}$ for $l_v \in \mathbb{R}, m_v \in \mathbb{Z}$

$\chi_v(z) = z^{-n_\sigma} \cdot \bar{z}^{-n_{c\sigma}}$

So $\begin{cases} 2l_v = -n_\sigma - n_{c\sigma} \\ m_v = -n_\sigma + n_{c\sigma} \text{ integer} \end{cases}$; say $n_\sigma \leq n_{c\sigma}$ ⟷ Hodge type $(\underset{\text{smaller}}{n_\sigma}, n_{c\sigma})$

$$L_v(\chi_v, s) = \Gamma_{\mathbb{C}}(s - n_\sigma) = 2(2\pi)^{-s+n_\sigma} \Gamma(s - n_\sigma)$$

Define $L(\chi, s) := \prod_v L_v(\chi_v, s)$

converges when $\mathrm{Re}(s) \gg 0$.

Examples (1) $F = \mathbb{Q}$, we have an isomorphism for any $N \in \mathbb{Z}_{>0}$

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0}^\times \simeq \hat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$$

So every Dirichlet character $\eta : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ induces a (finite) Hecke character

$$\eta_\mathbb{A} : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\eta} \mathbb{C}^\times$$

Exercise: $L(\eta_\mathbb{A}, s) = L(\eta^{-1}, s)$ $\qquad$ weight 0 story

(2) F number field, $Cl(F) = F^\times \backslash \mathbb{A}_{F,f}^\times / \hat{\mathcal{O}}_F^\times$.

Any character $\chi : Cl(F) \longrightarrow \mathbb{C}^\times$ defines a Hecke character

$$\chi_\mathbb{A} : F^\times \backslash \mathbb{A}_F^\times \longrightarrow F^\times \backslash \mathbb{A}_F^\times / F_\mathbb{R}^\times \cdot \hat{\mathcal{O}}_F^\times \simeq Cl(F) \xrightarrow{\chi} \mathbb{C}^\times.$$

(3) $|\cdot| : F^\times \backslash \mathbb{A}_F^\times \xrightarrow{Nm} \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \xrightarrow{|\cdot|} \mathbb{C}^\times$

$$(a_v)_v \longmapsto \prod |a_v|_v.$$

$L(|\cdot|, s) = \zeta_F(s+1)$ $\qquad$ weight $= -1$.

Tate's thesis: meromorphic continuation + functional equation:

$$L(\chi, s) = \varepsilon(\chi, s) \cdot L(\chi^{-1}, 1-s).$$

§2 Algebraic Hecke characters versus p-adic Hecke characters

Definition. A p-adic Hecke character is a continuous character $\chi^{\mathbb{Q}_p}: F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}_p^\times$

Due to incompatibility of topology, $\chi^{(p)}$ factors through $F^\times \backslash \mathbb{A}_F^\times / (F \otimes \mathbb{R})^{\times, 0}$

but for a p-adic place $v | p$, $\chi_v^{(p)}: F_v^\times \longrightarrow \mathbb{C}_p^\times$ can be quite interesting

Question: How to "compare" Hecke characters with p-adic Hecke characters?

$$\begin{array}{ccc} \{\text{All Hecke characters}\} & & \{\text{All } p\text{-adic Hecke characters}\} \\ \cup\!\mid & & \cup\!\mid \\ \{\text{Algebraic Hecke characters}\} & \xleftrightarrow[\sim]{\text{theorem}} & \{\text{Algebraic } p\text{-adic Hecke characters}\} \\ \cup\!\mid & & /\!\!/ \\ & \{\text{finite Hecke characters}\} & \end{array}$$

Definition: A Hecke character $\chi: F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ is called algebraic

if $\forall$ real embedding $\sigma \in \mathrm{Hom}(F, \mathbb{R})$, $\chi_\sigma = |\cdot|^{n_\sigma}$ or $\mathrm{sgn}(\cdot) \cdot |\cdot|^{n_\sigma}$ for $n_\sigma \in \mathbb{Z}$, and

$\forall$ complex embedding $\sigma \in \mathrm{Hom}(F, \mathbb{C})$, $\chi_\sigma(z) = \sigma(z)^{n_\sigma} \cdot c\sigma(z)^{n_{c\sigma}}$ for $n_\sigma, n_{c\sigma} \in \mathbb{Z}$.

The character χ is called finite if $\mathrm{Im}\,\chi$ is finite, or equivalently, all $n_\sigma = 0$.

Definition. A p-adic Hecke character $\chi: F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}_p^\times$ is called algebraic

if for any p-adic place v of F, for every $\tau \in \mathrm{Hom}(F_v, \mathbb{C}_p)$, there exists an integer n_τ

s.t. $$\begin{aligned} \mathcal{O}_{F_v}^\times &\longrightarrow \mathbb{C}_p^\times \\ x_v &\longmapsto \chi_v(x_v) \cdot \prod_{\tau \in \mathrm{Hom}(F_v, \mathbb{C}_p)} \tau(x_v)^{-n_\tau} \end{aligned}$$ is a character with finite image.

Remark: For an algebraic character χ, there are constraints on possible values of n_σ's

note: $\exists$ open subgroup $(1 + n\hat{\mathcal{O}}_F)^\times \subseteq \mathbb{A}_{F,f}^\times$ s.t. $\chi|_{1+n\hat{\mathcal{O}}_F^\times} = \mathrm{triv}$.

Thus, for every element $x \in \mathcal{O}_F^\times \cap (1 + n\hat{\mathcal{O}}_F)^\times =: \mathrm{Unit}_n$,

$$\underset{\substack{\uparrow \\ \text{global elements}}}{1} = \chi(x) = \prod_{\sigma: F \to \mathbb{C}} \sigma(x)^{-n_\sigma}$$

will discuss this later.

Theorem Let $\mathbb{Q}^{alg}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Fix an embedding $\mathbb{Q}^{alg} \overset{i}{\hookrightarrow} \mathbb{C}_p$

Then we get a 1:1 correspondence $\mathrm{Hom}(F,\mathbb{C}) \simeq \mathrm{Hom}(F,\mathbb{C}_p)$.

We have the following diagram

$$\begin{tikzcd}
\{\text{algebraic Hecke characters}\} & & \{\text{algebraic } p\text{-adic Hecke characters}\} \\
& \left\{ \text{continuous characters } \chi^{\mathbb{Q}} : \mathbb{A}_F^\times \to \mathbb{Q}^{alg,\times},\ \text{s.t.} \begin{array}{l} \forall x \in F^\times,\ \chi^{\mathbb{Q}}(x) = \prod_{\sigma \in \mathrm{Hom}(F,\mathbb{C})} \sigma(x)^{n_\sigma} \\ \forall v \text{ finite},\ \chi_v^{\mathbb{Q}}(\mathcal{O}_{F_v}^\times) \text{ is finite} \\ \forall v \text{ arch. } \chi_v^{\mathbb{Q}}|_{F_v^{\times,0}} \text{ is trivial} \end{array} \right\} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Proof: Given $\chi^{\mathbb{Q}}$, we define an algebraic Hecke charater $\chi^{\mathbb{R}}$

and an algebraic p-adic Hecke charater $\chi^{\mathbb{Q}_p}$, as follows.

$$\chi^{\mathbb{R}}\left((x_v)_v\right) := \chi^{\mathbb{Q}}\left((x_v)_v\right) \cdot \prod_{v|\infty} \prod_{\sigma : F_v \to \mathbb{C}} \sigma(x_v)^{-n_\sigma}$$

$$\chi^{\mathbb{Q}_p}\left((x_v)_v\right) := \chi^{\mathbb{Q}}\left((x_v)_v\right) \cdot \prod_{v|p} \prod_{\tau : F_v \to \overline{\mathbb{Q}}_p} \tau(x_v)^{-n_\tau}$$

Easy to check that $\forall x \in F^\times$, $\chi^{\mathbb{R}}(x) = 1$, $\chi^{\mathbb{Q}_p}(x) = 1$.

We need to show that $\chi^{\mathbb{Q}} \rightsquigarrow \chi^{\mathbb{R}}$ is a bijection to algebraic Hecke charaters

Conversely, given an algebraic Hecke charater $\chi^{\mathbb{R}} = \prod_v \chi_v^{\mathbb{R}} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$

<u>Claim</u>: For every finite place v of F, $\chi_v(F_v^\times) \subseteq \mathbb{Q}^{alg,\times}$

For topological reasons, $\chi_v(\mathcal{O}_{F_v}^\times) \subseteq \{\text{roots of unity}\}$.

For a uniformizer ϖ_v, note that $\mathfrak{p}_v \subseteq \mathcal{O}_F$ has finite order, say n_v in $Cl(\mathcal{O}_F)$

So, $\varpi_v^{n_v} \sim (\beta_v)$ is a principal ideal. for $\beta_v \in \mathcal{O}_F$

$$1 = \chi(\beta_v) = \prod_w \chi_w(\beta_v) = \prod_{\sigma : F \hookrightarrow \mathbb{C}} \sigma(\beta_v)^{n_\sigma} \cdot \chi_v(\beta_v) \cdot (\text{some roots of unity})$$

$$\Rightarrow \chi_v(\varpi_v) \in \mathbb{Q}^{alg,\times}$$

Now, we can safely define $\chi^{\mathbb{Q}}((x_v)_v) := \prod_{v\nmid\infty} \chi_v^{\mathbb{R}}(x_v) \cdot \prod_{v=\mathbb{R}} \mathrm{sgn}(\chi_v)(x_v)$. ✓

The comparison to algebraic p-adic character is similar. $\square$

Remark: {algebraic p-adic Hecke characters} can be far from dense in {all p-adic Hecke characters}

Issue 1: Leopoldt conjecture

$$\mathcal{O}_F^\times \otimes \mathbb{Q} \xrightarrow{\mathrm{reg}_p} \prod_{v\mid p} F_v \xrightarrow{\sum \mathrm{Tr}_{F_v/\mathbb{Q}_p}} \mathbb{Q}_p$$
$$x \longmapsto (\log_p(x_v))_v \longrightarrow 0$$

Then for a basis $u_1, \cdots u_{r_1+r_2-1}$ of $\mathcal{O}_F^\times \otimes \mathbb{Q}$,

$\mathrm{reg}(u_1), \cdots, \mathrm{reg}(u_{r_1+r_2-1})$ are $\mathbb{Q}_p$-linearly independent. (they are indeed $\mathbb{Q}$-linear indep for trivial reasons)

Issue 2: If F is not CM, there are far fewer algebraic Hecke characters.

Fancy writing: Consider the algebraic group $G := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$.

Let W be the following 1-dim'l representation $/\mathbb{Q}^{alg}$:

$$\rho : G_{\mathbb{Q}^{alg}} = \prod_{\sigma : F \hookrightarrow \mathbb{Q}^{alg}} \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

$$(x_\sigma)_\sigma \longmapsto \prod_\sigma x_\sigma^{n_\sigma}$$

Fix embeddings $\mathbb{C} \hookleftarrow \mathbb{Q}^{alg} \hookrightarrow \mathbb{C}_p$

Fix subgroups $K_v \subseteq G(F_v)$ s.t. if v is finite, K_v compact open

if $v \mid \infty$, $K_v = G(F_v)^\circ$

$K = \prod_v K_v$, $K^{(\infty)} := \prod_{v\nmid\infty} K_v$

$K^{(p)} := \prod_{v\nmid p} K_v$

$$\begin{tikzcd}
G(\mathbb{Q})\backslash G(\mathbb{A})/K^{(\infty)} \arrow[d, "/K_\infty"] \arrow[r, dash] & G(\mathbb{Q})\backslash G(\mathbb{A})/K^{(\infty)} \times W_{\mathbb{R}} \\
G(\mathbb{Q})\backslash G(\mathbb{A})/K \arrow[r, dash] & G(\mathbb{Q})\backslash G(\mathbb{A})/K^{(\infty)} \times^{K_\infty} W_{\mathbb{R}} =: \underline{{}_{K_\infty}W_{\mathbb{R}}}
\end{tikzcd}$$

$$\begin{tikzcd}
G(\mathbb{A})/K \arrow[d, "G(\mathbb{Q})\backslash"] \arrow[r, dash] & W_{\mathbb{Q}} \times G(\mathbb{A})/K \\
G(\mathbb{Q})\backslash G(\mathbb{A})/K \arrow[r, dash] & \underbrace{W_{\mathbb{Q}} \times}_{\text{turned into right action using } g^{-1}}{}^{G(\mathbb{Q})} G(\mathbb{A})/K =: \underline{W_{\mathbb{Q}, G(\mathbb{Q})}}
\end{tikzcd}$$

Similarly, define $\underline{{}_{K_p}W_{\mathbb{Q}_p}}$

an algebraic Hecke character χ belongs to here

$$\Gamma\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K,\ {}_{K_\infty}\underline{W_{\mathbb{R}}}\right) := \left\{ f^{\mathbb{R}}: G(\mathbb{Q})\backslash G(\mathbb{A})/K^{(\infty)} \to W_{\mathbb{R}},\ \text{s.t. } f(gg_\infty) = g_\infty^{-1} f(g)\right\}$$

$\simeq$ after $\otimes\mathbb{R}$ $\uparrow$

$$f^{\mathbb{R}}((g_v)) = g_\infty^{-1} f^{\mathbb{Q}}((g_v))$$

$$\uparrow$$

$$f^{\mathbb{Q}}$$

$$\downarrow$$

$$f^{\mathbb{Q}_p}((g_v)) = g_p^{-1} f^{\mathbb{Q}}((g_v))$$

$$\Gamma\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K,\ \underline{W_{\mathbb{Q}}}_{G(\mathbb{Q})}\right) := \left\{ f^{\mathbb{Q}}: G(\mathbb{A})/K \to W_{\mathbb{Q}},\ \text{s.t. } f(\gamma g) = \gamma f(g)\right\}$$

$\simeq$ after $\otimes\mathbb{Q}_p$ $\downarrow$

$$\Gamma\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K,\ {}_{K_p}\underline{W_{\mathbb{Q}_p}}\right) := \left\{ f^{\mathbb{Q}_p}: G(\mathbb{Q})\backslash G(\mathbb{A})/K^{(p)} \to W_{\mathbb{Q}_p},\ \text{s.t. } f(gg_p) = g_p^{-1} f(g)\right\}$$

<u>Remark</u>: Fancy writing does not need G to be a torus, can also do $H^i(-)$.

§3 <u>Possible weights for algebraic Hecke characters</u>

<u>Question</u>. Let F be a number field. What are the constraint on n_σ's for an algebraic Hecke character?

<u>Case 1</u>: F is totally real, i.e. all $\mathbb{C}$-embeddings are $\mathbb{R}$-embeddings.

(Exercise: Composites of totally real fields are totally real.
So the Galois closure of totally real field is totally real.)

$\mathcal{O}_F^\times \cap (1+n\hat{\mathcal{O}}_F)^\times$ is free of rank $[F:\mathbb{Q}]-1$ by Dirichlet unit theorem

$$\begin{array}{ccc} \mathcal{O}_F^\times \cap (1+n\hat{\mathcal{O}}_F)^\times & \xrightarrow{\log|\sigma(\cdot)|} & \left(\prod_{\sigma:F\to\mathbb{R}} \mathbb{R}\right)^{\text{Sum}=0} \\ \Big\downarrow x\mapsto \prod_\sigma \sigma(x)^{n_\sigma} & & \Big\downarrow \sum_\sigma n_\sigma\cdot(-) \\ 1 & \xrightarrow{\log} & 0 \end{array}$$

image = lattice

$\Rightarrow$ all n_σ are equal.

<u>Case 2</u>: F is a CM field, i.e., a quadratic extension of a totally real field F^+
that is an extension $\mathbb{C}/\mathbb{R}$ over every archimedean place of F.

(<u>Exercise</u>: Composites of CM fields are CM,
In particular, the Galois closure of a CM field is CM.
If $F/\mathbb{Q}$ is Galois, there's only one complex conjugation c; it's in the center of $\mathrm{Gal}(F/\mathbb{Q})$.)

By Dirichlet unit theorem, $\mathcal{O}_{F^+}^\times$ has finite index in $\mathcal{O}_F^\times$.

So the essential constraint is that for all $\sigma \in \mathrm{Hom}(F, \mathbb{C})$,

$$n_\sigma + n_{c\sigma} \text{ is independent of } \sigma.$$

<u>General case</u>:

<u>Note</u>: Inside any field F, there's a maximal CM subfield F_{CM}.

Then
$$\begin{tikzcd} n : \mathrm{Hom}(F, \mathbb{C}) \arrow[r, "\sigma \mapsto n_\sigma"] \arrow[dr] & \mathbb{Z} \\ & \mathrm{Hom}(F_{CM}, \mathbb{C}) \arrow[u, dashed] \end{tikzcd}$$
factors through and $n_\sigma + n_{c\sigma}$ is indep of σ