Special values of L-functions 15

—— Modular forms and their periods

§1. Recollection of facts on tori

Definition. A torus over $\mathbb{Q}$ is a group scheme $T/\mathbb{Q}$ s.t. $T_{\bar{\mathbb{Q}}} \simeq \mathbb{G}_{m,\bar{\mathbb{Q}}}^n$ for some $n \in \mathbb{Z}_{\geq 0}$

Define group of characters $X^*(T) := \mathrm{Hom}_{\bar{\mathbb{Q}}}(T_{\bar{\mathbb{Q}}}, \mathbb{G}_{m,\bar{\mathbb{Q}}}) \simeq \mathbb{Z}^n$, $\circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$

group of cocharacters $X_*(T) := \mathrm{Hom}_{\bar{\mathbb{Q}}}(\mathbb{G}_{m,\bar{\mathbb{Q}}}, T_{\bar{\mathbb{Q}}}) \simeq \mathbb{Z}^n$.

There's a perfect pairing $X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$

$$(\chi, \eta) \longmapsto m, \text{ s.t. } \begin{tikzcd} \mathbb{G}_m \arrow[r, "\eta"] \arrow[rr, bend left, "x \mapsto x^m"] & T \arrow[r, "\chi"] & \mathbb{G}_m \end{tikzcd}$$

Fact: $\exists$ a 1-1 correspondence

$$\{\text{Tori over } \mathbb{Q}\} \longleftrightarrow \{\text{finite free } \mathbb{Z}\text{-modules with continuous } \mathrm{Gal}_{\mathbb{Q}}\text{-action}\}$$

$$T \longmapsto X^*(T)$$

$$\mathrm{Spec}\left(\mathbb{Q}^{alg}[M]^{\mathrm{Gal}_{\mathbb{Q}}}\right) \longleftarrow M$$ (can view each $\eta \in X^*(T)$ as a function on $T_{\bar{\mathbb{Q}}}$)

isogeny $S \twoheadrightarrow T \longleftrightarrow X^*(T) \hookrightarrow X^*(S)$

ker finite of size m — coker of size m

Subtorus $S \hookrightarrow T \longleftrightarrow X^*(T) \twoheadrightarrow X^*(S)$

* If $M_{\mathbb{Q}} = M'_{\mathbb{Q}} \oplus M''_{\mathbb{Q}}$ as Galois modules over $\mathbb{Q}$

$\rightsquigarrow M' := M'_{\mathbb{Q}} \cap M$ and $M'' := M''_{\mathbb{Q}} \cap M$, then $M' \oplus M'' \hookrightarrow M$ finite index

$\Rightarrow T_{M'} \times T_{M''} \twoheadrightarrow T_M$ isogeny.

Theorem. For any open compact subgroup $K_T \subseteq T(\mathbb{A}_f)$, modulo torsions,

$$T(\mathbb{Q}) \cap K_T \text{ is a free } \mathbb{Z}\text{-module of rank} = \mathrm{rank}\, X^*(T)^{c=1} - \mathrm{rank}\, X^*(T)^{\mathrm{Gal}_{\mathbb{Q}}}$$

Proof: Using the decomposition above, may assume that $X^*(T)_{\mathbb{Q}}$ is an irreducible rep'n of $\mathrm{Gal}_{\mathbb{Q}}$

Case 1: $X^*(T)_{\mathbb{Q}}$ is the trivial rep'n $\Rightarrow T = \mathbb{G}_m$. so $\mathbb{G}_m(\mathbb{Q}) \cap K_{\mathbb{G}_m}$ is finite.

<u>Case 2</u>: $X^*(T)_{\mathbb{Q}}$ is not the trivial rep'n. $\circlearrowleft \mathrm{Gal}_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q})$

$$\rightsquigarrow \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})] \longrightarrow X^*(T) \quad \text{finite cokernel.}$$

$$\longleftrightarrow T \longrightarrow \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m F}$$

$$\begin{array}{ccc} T(\mathbb{Q}) \cap K_T & \longrightarrow & X^*(T)_{\mathbb{R}}^{c=1} \leftarrow \text{not the trivial sub} \\ \downarrow & & \cap\mid \\ \mathcal{O}_F^\times = (\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m)(\mathbb{Q}) \cap \hat{\mathcal{O}}_F^\times & \xrightarrow{\prod_\sigma \log|\sigma(\cdot)|} & (\mathbb{R}^{r_1(F)+r_2(F)})^{\mathrm{Sum}=0} = \mathbb{R}[\mathrm{Gal}(F/\mathbb{Q})]^{\mathrm{Sum}=0}_{c=1} \end{array}$$

$- \otimes_{\mathbb{Z}} \mathbb{R}$ becomes isom.

$$\Rightarrow \mathrm{rank}\left(T(\mathbb{Q}) \cap K_T\right) = \mathrm{rank}\, X^*(T)_{\mathbb{R}}^{c=1} \text{ in this case} \qquad \square$$

<u>Corollary</u>. Given a torus $T/\mathbb{Q} \rightsquigarrow X^*(T)_{\mathbb{Q}}$.

Under the Galois action, $X^*(T)_{\mathbb{Q}} = X_{0,\mathbb{Q}} \oplus X_{CM,\mathbb{Q}} \oplus X_{\text{non-CM},\mathbb{Q}}$ as $\mathbb{Q}[\mathrm{Gal}_{\mathbb{Q}}]$-modules

where $X_{0,\mathbb{Q}} = X^*(T)_{\mathbb{Q}}^{\mathrm{Gal}_{\mathbb{Q}}}$

$X_{CM,\mathbb{Q}} = \bigoplus$ subrep'ns on which c acts by scalar -1.

$X_{\text{non-CM},\mathbb{Q}} = \bigoplus$ all other irreducible components

Put $X_0 := \mathrm{Im}\left(X^*(T) \to X_{0,\mathbb{Q}}\right)$ and define X_{CM}, $X_{\text{non-CM}}$ similarly

Then $X^*(T) \to X_0 \oplus X_{CM} \oplus X_{\text{non-CM}} \longleftrightarrow T_0 \times T_{CM} \times T_{\text{non-CM}} \twoheadrightarrow T$

Then for any open compact subgroup $K_T \subseteq T(\mathbb{A}_f)$,

the Zariski closure of $T(\mathbb{Q}) \cap K_T$ is $\mathrm{Im}(T_{\text{non-CM}} \to T)$.

Proof: Every nontriv irreducible subtorus of T $\longleftrightarrow$ an irreducible subrep'n of $X^*(T)_{\mathbb{Q}}$

$T_W \qquad\qquad W$

If $\mathrm{rank}\, X^*(T_W)_{\mathbb{Q}}^{c=1} \neq \mathrm{rank}\, X^*(T_W)_{\mathbb{Q}}^{\mathrm{Gal}_{\mathbb{Q}}=1}$, then $\forall$ open compact subgroup $K_W \subseteq T_W(\mathbb{A}_f)$

$T_W(\mathbb{Q}) \cap K_W$ has $\neq 0$ rank

$\Rightarrow$ its Zariski closure is ≥ 1-dim'l and contains a subtorus

But T_W is irred $\Rightarrow T_W(\mathbb{Q}) \cap K_W$ is Zariski dense. $\square$

§2. Algebraic Hecke character.

Let F be a number field. An algebraic Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ is a cont. char

s.t. $\forall v = \mathbb{R} \leftrightarrow \sigma : F \to \mathbb{R}$, $\chi_v(x_v) = |x_v|^{-n_\sigma}$ or $|x_v|^{-n_\sigma}\operatorname{sgn}(x_v)$ for $n_\sigma \in \mathbb{Z}$

$\forall v = \mathbb{C} \leftrightarrow \sigma, c\sigma : F \to \mathbb{C}$, $\chi_v(x_v) = \sigma(x_v)^{-n_\sigma} \cdot c\sigma(x_v)^{-n_{c\sigma}}$ for $n_\sigma, n_{c\sigma} \in \mathbb{Z}$.

Think of n_σ's as a function $n = n_\chi : \operatorname{Hom}(F, \mathbb{C}) \to \mathbb{Z}$.

Let $F_{CM} :=$ max'l CM subfield of F.

Theorem: n_χ factors as a function $\operatorname{Hom}(F, \mathbb{C}) \to \operatorname{Hom}(F_{CM}, \mathbb{C}) \xrightarrow{n_{CM}} \mathbb{Z}$

& $n_{CM,\sigma} + n_{CM,c\sigma}$ is independent of σ.

Proof: Consider $T = \operatorname{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$. A constraint on n_σ's is that:

Thinking n as a function in $\mathcal{O}[T_\mathbb{C}]^\times = \mathcal{O}\left[\prod_{\sigma : F \to \mathbb{C}} \mathbb{G}_m\right]^\times$

$$n : (t_\sigma)_\sigma \mapsto \prod t_\sigma^{n_\sigma}$$

it is trivial on $\mathcal{U}nit_\mathfrak{n} := T(\mathbb{Q}) \cap (1 + \mathfrak{n}\hat{\mathcal{O}}_F)^\times$ for some ideal $\mathfrak{n} \subseteq \mathcal{O}_F$

So n is trivial on the Zariski closure of $T(\mathbb{Q}) \cap (1 + \mathfrak{n}\hat{\mathcal{O}}_F)^\times$ in $T_\mathbb{C}$

By corollary above, we need to study $\mathbb{Q}[\operatorname{Hom}(F, \mathbb{C})] \circlearrowleft \operatorname{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

Clearly, $\mathbb{Q}[\operatorname{Hom}(F, \mathbb{C})]^{\operatorname{Gal}_\mathbb{Q}} = \mathbb{Q}$ 1-dim'l

If $W \subseteq \mathbb{Q}[\operatorname{Hom}(F, \mathbb{C})]$ is an irreducible sub-$\operatorname{Gal}_\mathbb{Q}$-module,

say factoring exactly through $\operatorname{Gal}(E/\mathbb{Q})$.

s.t. $c = -1$ on W (Think of $c \in \operatorname{Gal}(E/\mathbb{Q})$)

$\Rightarrow c$ commutes with all elements in $\operatorname{Gal}(E/\mathbb{Q}) \Rightarrow c \in Z(\operatorname{Gal}(E/\mathbb{Q}))$

$\Rightarrow E$ is a totally imaginary ext'n of $E^{c=1}$ ~ totally real $\Rightarrow E$ CM.

Condition $\Rightarrow W \subseteq \mathbb{Q}[\operatorname{Hom}(F, \mathbb{C})]^{\operatorname{Gal}_E} = \mathbb{Q}[\operatorname{Hom}(E \cap F, \mathbb{C})]$.

$$\begin{tikzcd}
E \arrow[d, dash] \arrow[dr, dash] & & F \arrow[dl, dash] \\
E^+ \arrow[dr, dash] & E\cap F \arrow[d, dash] & \\
& E^+\cap F \arrow[d, dash] & \\
& \mathbb{Q} &
\end{tikzcd}$$

Exercise: $E^+\cap F$ is totally real & $[E\cap F : E^+\cap F] \leq 2$.

So $E\cap F$ is either totally real or CM.

not possible b/c c acts by 1 here.

So $W \subseteq \mathbb{Q}[\mathrm{Hom}(F_{CM}, \mathbb{C})]$ $\square$

§3 Critical values for algebraic Hecke characters

$\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ algebraic Hecke character of wt $n : \mathrm{Hom}(F,\mathbb{C}) \to \mathbb{Z}$

s.t. for $v|\infty$, $\chi_v(x_v) = \begin{cases} |x_v|^{-n_\sigma} \text{ or } \mathrm{sgn}(x_v)\cdot|x_v|^{-n_\sigma} & v \leftrightarrow \sigma\ \mathbb{R}\text{-embedding} \\ \sigma(x_v)^{-n_\sigma}\cdot c\sigma(x_v)^{-n_{c\sigma}} & v \leftrightarrow \sigma : F_v \to \mathbb{C}. \end{cases}$

Next lecture: $\chi \leftrightarrow M_\chi$ motive attached to χ defined over F

For Deligne's conj $\rightsquigarrow$ need to view M_χ as a motive$/\mathbb{Q}$: $M_{\chi,\mathbb{Q}}$

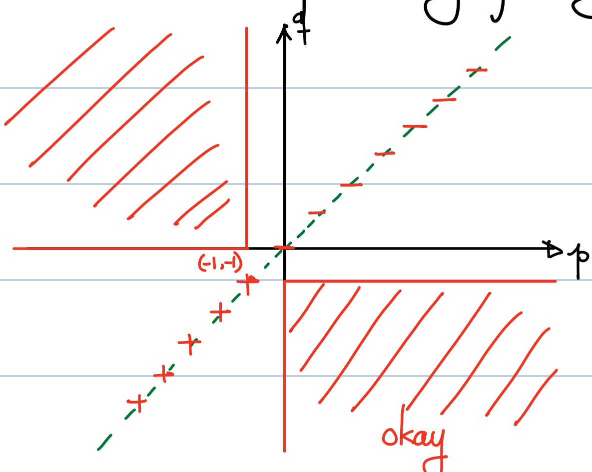
Hodge type of χ are precisely (n_σ, n_σ) for each $\sigma : F \hookrightarrow \mathbb{R}$

$(n_\sigma, n_{c\sigma})$ and $(n_{c\sigma}, n_\sigma)$ for each $\sigma : F \hookrightarrow \mathbb{C}$ complex.

& corresponding sign on $M_{\chi,\mathbb{Q},\mathrm{Betti}}$ is $\mathrm{sgn}(\cdot)|\cdot|^{-n_\sigma} \leftrightarrow (-1)^{n_\sigma+1}$, $|\cdot|^{-n_\sigma} \leftrightarrow (-1)^{n_\sigma}$

each complex embedding get one $+$ sign and one $-$ sign.

Recall the if $L(\chi, 0)$ is critical if and only if Hodge type belongs to



Case 1: F does not contain a CM subfield $\Rightarrow$ all $n_\sigma = n$.

Then if $L(\chi,0)$ is critical, then we are "on the diagonal" (n,n)

$\to$ can't have a complex embedding b/c it will produce both signs $\Rightarrow$ F is totally real

can only have real embedding $\rightsquigarrow$ $n \geq 0$ every $v = \mathbb{R}$, $\chi_v(x_v) = \mathrm{sgn}(x_v)^{n+1} |x_v|^{-n} = -x_v^{-n}$

$n \leq -1$ every $v = \mathbb{R}$, $\chi_v(x_v) = \mathrm{sgn}(x_v)^n \cdot |x_v|^{-n} = x_v^{-n}$

Case 2: F contains a max'l CM subfield F_{CM}. $n: \mathrm{Hom}(F,\mathbb{C}) \to \mathrm{Hom}(F_{CM},\mathbb{C}) \to \mathbb{Z}$.

Condition: $n = n_\sigma + n_{c\sigma}$ indep of σ

Then if $L(\chi,0)$ is critical for every σ, $\max\{n_\sigma, n_{c\sigma}\} \geq 0$ and $\min\{n_\sigma, n_{c\sigma}\} \leq -1$.

§4 Motives attached to algebraic Hecke characters

F number field, for a finite Hecke character $\chi: F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$

one can construct a Galois rep'ns $\mathrm{Gal}_F^{ab} \xrightarrow{\sim} F^\times \backslash \mathbb{A}_F^\times / F_\mathbb{R}^{\times,0} \xrightarrow{\chi} \mathbb{Q}^{alg,\times}$

This corresponds to a motive that appear in $\mathrm{Spec}\, F^{ab}$.

In general, given $\chi^\mathbb{Q}: \mathbb{A}_F^\times \longrightarrow \mathbb{Q}^{alg,\times}$ s.t. $\forall \gamma \in F$, $\chi^\mathbb{Q}(\gamma) = \prod_{\sigma: F \to \mathbb{C}} \sigma(\gamma)^{n_\sigma}$

Let K be the coefficient field of $\chi^\mathbb{Q}$

want a motive M_χ s.t. $M_{\chi,et,\ell} \simeq \chi^{\mathbb{Q}_\ell}$ coefficients in $K \otimes \mathbb{Q}_\ell$

$M_{\chi,dR} \simeq \chi^\mathbb{R}$ coefficients in $K \otimes \mathbb{R}$

- Suffices to treat the "algebraic part" of Hecke character.

* CM abelian variety: Let K be a CM field with c the complex conjugation s.t. $K^+ := K^{c=1}$ is totally real.

- A CM type Φ is a subset $\Phi \subseteq \mathrm{Hom}(F,\mathbb{C})$ s.t. $\mathrm{Hom}(K,\mathbb{C}) = \Phi \sqcup c\Phi$

i.e. for each $\mathbb{R}$-embedding $\tau: K^+ \to \mathbb{R}$, there are exactly two embeddings $\sigma, c\sigma : K \to \mathbb{C}$ extending τ.

- We identify $K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma \in \Phi} \mathbb{C}_\sigma$ Here we made the choice between σ and $c\sigma$

$$\begin{array}{c} K \otimes_{\mathbb{Q}} \mathbb{R} \\ \cup\!\shortmid \\ \mathcal{O}_K \end{array}$$

$\rightsquigarrow A_\Phi := \left(\prod_{\sigma \in \Phi} \mathbb{C}_\sigma\right) / \mathcal{O}_K$ so that $\mathrm{Tang}_0 A_\Phi \simeq \prod_{\sigma\in\Phi} \mathbb{C}_\sigma$

$\circlearrowleft \mathcal{O}_K$ i.e. $\mathcal{O}_F$ acts on the tangent space having eigenvalues $\sigma \in \Phi$.

$H_1(A_\Phi, \mathbb{Z}) \simeq \mathcal{O}_K$

$\xrightarrow{\otimes_{\mathbb{Z}} \mathbb{Q}}$ This is a motive with coeffs in K

<u>Polarization</u>. Pick a purely imaginary element $\delta \in \mathcal{O}_K^{c=-1}$ s.t. under $\forall \sigma \in \Phi$, $\sigma(\delta) \in \mathbb{R}_{>0} \cdot i$

$\psi : \mathcal{O}_K \times \mathcal{O}_K \longrightarrow 2\pi i \mathbb{Z}$

$\psi(x,y) = -2\pi i \cdot \mathrm{Tr}_{K/\mathbb{Q}}(x\delta\bar{y})$ symplectic

(Note : $(x,y) \mapsto \frac{1}{2\pi i}\psi(x, h(i)\bar{y}) = \sum_{\sigma\in\Phi} -\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(x\cdot\delta i\cdot\bar{y})$ is positive definite.

<u>Definition</u>. Consider $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C}) \circlearrowleft \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

$\rightsquigarrow \mathrm{Stab}(\Phi) \subseteq \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$ stabilizer as a subset

Define the <u>reflex field of Φ</u> to be $K_\Phi = (\mathbb{Q}^{alg})^{\mathrm{Stab}(\Phi)}$

<u>Fact</u>: If A_Φ has CM by $\mathcal{O}_K$, then A_Φ is def'd over the Hilbert class field of K_Φ

But can view A_Φ as a K_Φ-scheme to produce motives.

<u>Below is a little strange</u> ???

To construct motives attached to $\chi^{\mathbb{Q}} : \mathbb{A}_F^\times / F_{\mathbb{R}}^\circ \longrightarrow K^\times$

$\rightsquigarrow$ it will come from an abelian variety with CM by $\mathcal{O}_K$, and defined $/F$

So we need $F = K_\Phi$.

Interesting duality: F CM & Galois

$$\begin{tikzcd}
n : \mathrm{Hom}(F, \bar{\mathbb{Q}}) = \mathrm{Hom}(F, F^{Gal}) \arrow[r] & \mathbb{Z} \\
\mathrm{Hom}(F^{Gal}, F^{Gal}) \arrow[u, two heads] &
\end{tikzcd}$$

$\updownarrow$ Φ_F a CM type for F

$\rightsquigarrow$ induces a CM type for F^{Gal}

Note: $\mathrm{Gal}(F^{Gal}/\mathbb{Q}) \circlearrowright \mathrm{Hom}(F^{Gal}, F^{Gal}) \circlearrowleft \mathrm{Gal}(F^{Gal}/\mathbb{Q})$

$$\begin{tikzcd}
 & \Phi \text{ CM type} \subseteq \mathrm{Hom}(F^{Gal}, F^{Gal}) \arrow[dl, "\pi_{left}"'] \arrow[dr, "\pi_{right}"] & \\
\Phi_F \subseteq \mathrm{Hom}(F, F^{Gal}) & & \Phi_K \subseteq \overleftarrow{\mathrm{Map}}(F^{Gal}, K)
\end{tikzcd}$$

$c \in \mathrm{Gal}(F^{Gal}/\mathbb{Q})$ (on both sides)

$n \rightsquigarrow \Phi_F \mapsto \Phi_K$ for K.

Take an abel var A CM by K w/ CM type Φ_K, defined over F.

$$\rightsquigarrow \mathrm{Gal}_F^{ab} \longrightarrow GL_1(H_1^{et}(A_{\bar{\mathbb{Q}}^{alg}}, \mathbb{Q}_\ell))$$

$$\mathrm{Gal}_F^{ab} \cong F^\times \backslash \mathbb{A}_F^\times / F_{\mathbb{R}}^{\times,0}$$