Special values of L-functions 17

—— Periods of algebraic Hecke characters

§1. Main results (of Kings and Sprang)

Recall: Deligne's conjecture for algebraic Hecke characters is divided into two parts

* F totally real, $\chi = \rho \cdot (|\cdot|^{-n} \circ N_{F/\mathbb{Q}})$ for ρ a finite character of $F^\times \backslash \mathbb{A}_F^\times / F_{\mathbb{R}}^{\times,0}$

  s.t. the sign at each $v = \mathbb{R}$ of F is $\begin{cases} -x_v^{-n} & n \geq 0 \\ x_v^{-n} & n \leq -1 \end{cases}$

  $L(\rho, -n)$ $\parallel$

  In this case, expect $L(\chi, 0) \in (2\pi i)^{-n} \cdot \mathbb{Q}^{alg}$

  (can be proved using Eisenstein series)

* K = CM field, L a finite extension of K

* $\chi = \rho \cdot (\chi_0 \circ N_{L/K})$ for ρ a finite character of $L^\times \backslash \mathbb{A}_L^\times$

  χ_0 an algebraic Hecke character

  $\chi_0 : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ s.t. $\forall v = \mathbb{C} \leftrightarrow \sigma, c\sigma \in \mathrm{Hom}(K, \mathbb{C})$ $\chi_{0,v}(x_v) = \sigma(x_v)^{-n_\sigma} c\sigma(x_v)^{-n_{c\sigma}}$

  χ critical if for any pair $\sigma, c\sigma$, $\max\{\sigma, c\sigma\} \geq 0$, $\min\{\sigma, c\sigma\} \leq -1$

  $\rightsquigarrow$ defines a CM type Φ of K $\Phi = \{\sigma \mid \sigma \leq -1\}$

  write $n \in \mathrm{Fun}(\mathrm{Hom}(K,\mathbb{C}) \to \mathbb{Z})$ as $n = \beta_0 - \alpha_0$ for $\beta_0 \in \mathrm{Fun}(\bar{\Phi} \to \mathbb{Z}_{\geq 0})$ $\alpha_0 \in \mathrm{Fun}(\Phi \to \mathbb{Z}_{\geq 0})$

* Let $A/\mathbb{Q}^{alg}$ be an abelian variety with CM by K

$$0 \to \underbrace{H^0(A, \Omega^1_A)}_{\text{free of rk1 over } K \otimes \mathbb{Q}^{alg}} \to H^1_{dR}(A/\mathbb{Q}^{alg}) \to H^1(A, \mathcal{O}_A)$$

$$H^1_{dR}(A/\mathbb{Q}^{alg}) = \underbrace{H_1(A(\mathbb{C}), \mathbb{Z})^\vee}_{=\,\mathfrak{a} \subseteq \mathcal{O}_K} \otimes \mathbb{Q}^{alg}$$

pick a basis ω

$\mathfrak{a} \subseteq \mathcal{O}_K$ for a fractional ideal $\leftarrow$ a K-basis ξ

$$\langle \omega, \xi \rangle =: \Omega \in \mathbb{C}^{\Phi}$$

dually, have a pairing $H^0(A^\vee, \Omega^1_{A^\vee}) \times H_1(A^\vee(\mathbb{C}), \mathbb{Q}) \longrightarrow \mathbb{C}^{c\Phi}$

$$\begin{array}{ccc} \omega^\vee & , & \xi^\vee \end{array} \mapsto \Omega^\vee \in \mathbb{C}^{c\Phi}$$

Rmk: $A^\vee$ has type $c\Phi$.

Theorem: $L(\chi, 0) \in \dfrac{(2\pi i)^{|\beta|}}{\Omega^{\alpha_0}\Omega^{\vee\beta_0}} \cdot \mathbb{Q}^{alg,\times}$ note: $\Omega^\vee \in \dfrac{(2\pi i)}{\bar{\Omega}} \cdot \mathbb{Q}^{alg,\times}$

§2 a p-adic version.

Setup: K CM field, Φ = CM type. Fix an embedding $\mathbb{Q}^{alg} \hookrightarrow \mathbb{C}_p$

$\Phi \rightsquigarrow \Phi_p \subseteq \mathrm{Hom}(K, \mathbb{C}_p)$, Φ_p^c

Assumption. (p-ordinary) Every p-adic place v of K^+ splits in K.

and can assign w and w^c s.t. all p-adic embeddings of $K_w \subseteq \Phi_p$

————— $K_{w^c} \subseteq \Phi_p^c$

$$\begin{array}{cc} K & w \quad w^c \\ | & \diagdown\ \diagup \\ K^+ & v \end{array}$$

$\Sigma_p = \{w\}$, $\Sigma_p^c = \{w^c\}$

Under this assumption, the CM abelian variety $A/\mathbb{Q}^{alg} \hookrightarrow \mathbb{C}_p$ is ordinary at p

Then we have $A[p^\infty] \simeq A[\Sigma_p^\infty] \oplus A[\Sigma_p^{c,\infty}]$

$$A[\Sigma_p^\infty] \underset{\alpha}{\cong} \mu_{p^\infty} \otimes \left(\bigoplus_w \mathcal{O}_{K_w}\right)$$

$$H^0(A, \Omega^1_{A/\mathbb{Q}^{alg}}) \otimes \mathbb{C}_p \overset{\alpha}{\simeq} H^0\left(\hat{\mathbb{G}}_m \boxtimes \bigoplus_w \mathcal{O}_{K_w}\right) \otimes \mathbb{C}_p$$

$$\omega \qquad\qquad \omega_{can,p} = \frac{dT}{1+T}$$

$\alpha(\omega) = \Omega_p \cdot \omega_{can,p}$ for $\Omega_p \in \mathbb{C}_p^{\Phi_p}$ with Ω_p well-def'd up to $\left(\bigoplus_w \mathcal{O}_{K_w}\right)^\times$

Theorem. Fix an open compact subgroup $K_f^{(p)} \subseteq \hat{\mathcal{O}}_L^{\times,(p)}$,

infinite abelian extension $L(p^\infty \cdot K_f^{(p)}) \longleftrightarrow$ subgroup $K_f^{(p)} \cdot L^\times \subseteq L^\times \backslash \mathbb{A}^\times_{L,f} \simeq \mathrm{Gal}_L^{ab}$

then there exists a p-adic measure μ on $\mathrm{Gal}(L(p^\infty \cdot K_f^{(p)})/L)$ such that

$\forall$ critical algebraic Hecke char χ of CM type Σ and infinite type $\mu = \beta - \alpha$

and conductor $K_f^{(p)}$, we have

$$\frac{1}{\Omega_p^{\alpha}\Omega_p^{\vee\beta}}\int_{\mathrm{Gal}(L(p^\infty K_f^{(p)})/L)} \chi(g)\,d\mu(g) = \frac{(\alpha-1)!\,(2\pi i)^{|\beta|}}{\Omega^{\alpha}\Omega^{\vee\beta}}\,(\text{local factors})\,L(\chi,0).$$

<u>Hope</u>: Explain some ideas of Katz's proof of this when L is imaginary quadratic.

§3. A quick introduction to modular forms

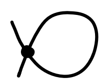
Fix $N \geq 4$.

Let $Y_1(N)$ be the moduli space of the functor

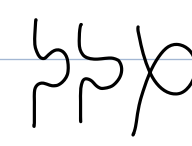
$$\mathrm{Sch}^{\mathrm{loc.noe}}/\mathbb{Z}[\tfrac{1}{N}] \longrightarrow \mathrm{Sets}$$

$$S \longmapsto \left\{ \begin{array}{l} E/S \text{ elliptic curve} \\ i : \mu_{N,S} \hookrightarrow E \end{array} \right\}$$

$$\begin{tikzcd} (\mathcal{E}, i) \arrow[d] & \mathcal{E}^{sm} \subseteq \overline{\mathcal{E}} \arrow[d, "\pi"] & \mathcal{E}^{sm}_x = \mathbb{G}_m \subseteq \overline{\mathcal{E}}_x \arrow[d] \\ Y_1(N) \arrow[u, bend left, "i"] \arrow[r, hook] & X_1(N) & \supseteq C = \text{cusp} \ni x \end{tikzcd}$$

$\overline{\mathcal{E}}_x =$  $\leftarrow$ folding up 0 & ∞

$i : \mu_N \hookrightarrow \mathbb{G}_m$



C: finite étale $/\mathbb{Z}[\frac{1}{N}]$.

Define $\omega := i^*\Omega^1_{\mathcal{E}^{sm}/X_1(N)}$

<u>Definition</u>. The space of weight k level $\Gamma_1(N)$ modular form is

$$S_k(\Gamma_1(N)) := H^0(X_1(N), \omega^k)$$

$$M_k(\Gamma_1(N)) := H^0(X_1(N), \omega^k(D))$$

<u>Note</u>: $\overline{\mathcal{E}}$ is itself smooth $/\mathbb{Z}[\frac{1}{N}]$ but not smooth over $X_1(N)$

Write $D := \pi^{-1}(\text{cusps})$; it is a divisor with simple normal crossing.

i.e. étale locally, $D \subseteq \overline{\mathcal{E}}$ is of the form

$$\mathrm{Spec}\,\mathbb{Z}[\tfrac{1}{N}][x_1,\dots,x_n]/(x_1,\dots x_m) \hookrightarrow \mathrm{Spec}\,\mathbb{Z}[\tfrac{1}{N}][x_1,\dots,x_n] \qquad (*)$$

* π is not smooth, but it is "log-smooth",

$$\Rightarrow\ 0 \to \pi^*\Omega^1_{X_1(N)/\mathbb{Z}[\frac{1}{N}]}(\log C) \to \Omega^1_{\bar{\mathcal{E}}/\mathbb{Z}[\frac{1}{N}]}(\log D) \to \underbrace{\Omega^1_{(\bar{\mathcal{E}},D)/(X_1(N),C)}}_{\text{quotient is locally free.}} \to 0$$

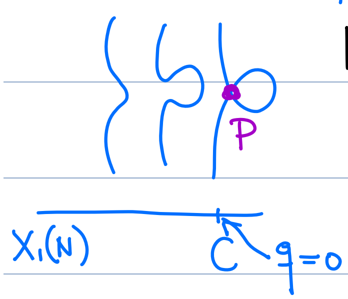
(Here, for $D \subseteq X/k$ a simple normal crossing divisor

$\Omega^1_{X/k}(\log D)$ is locally (for $(*)$)

$$\mathcal{O}_X\frac{dx_1}{x_1} \oplus \cdots \oplus \mathcal{O}_X\frac{dx_m}{x_m} \oplus \mathcal{O}_X dx_{m+1} \oplus \cdots \oplus \mathcal{O}_X dx_n\ )$$

<u>Remark</u>: $\Lambda^{\dim X}(\Omega^1_{X/k}(\log D)) = (\Lambda^{\dim X}\Omega^1_{X/k})(D)$.

Let's make a computation over the cusp to see $\Omega^1_{(\bar{\mathcal{E}},D)/(X_1(N),C)}$ is locally free.



locally near P, this map looks like

$$\begin{array}{ccc} \mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}][x,y,q]/\underset{R}{\underbrace{(xy-q)}} & \supseteq & Z(xy=0) \\ \downarrow & & \\ \mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}][q] & \supseteq & Z(q=0) \end{array}$$

We compute $\quad \mathrm{Coker}\left(R\cdot\frac{dq}{q} \to R\cdot\frac{dx}{x} \oplus R\cdot\frac{dy}{y}\right)$

$$q = xy \Rightarrow \frac{dq}{q} = \frac{dx}{x} + \frac{dy}{y}$$, So coker is locally free of rank 1.

Some simplification of notations: $\Omega^1_{X,\log} := \Omega^1_{X_1(N)/\mathbb{Z}[\frac{1}{N}]}(\log C)$

$\Omega^1_{\bar{\mathcal{E}},\log} := \Omega^1_{\bar{\mathcal{E}}/\mathbb{Z}[\frac{1}{N}]}(\log D)$, $\Omega^1_{\bar{\mathcal{E}}/X,\log} := \Omega^1_{(\bar{\mathcal{E}},D)/(X,C)}$

* There is a relative log-de Rham cohomology:

$$R\pi_*\left(\mathcal{O}_{\bar{\mathcal{E}}} \xrightarrow{d} \Omega^1_{\bar{\mathcal{E}}/X,\log}\right)$$

<u>Fact</u>: $R^1\pi_*\left(\mathcal{O}_{\bar{\mathcal{E}}} \xrightarrow{d} \Omega^1_{\bar{\mathcal{E}}/X,\log}\right)$ is locally free of rank 2.

Denote $H^1_{dR,\log}(\bar{\mathcal{E}}/X) := R^1\pi_*\left(\mathcal{O}_{\bar{\mathcal{E}}} \xrightarrow{d} \Omega^1_{\bar{\mathcal{E}}/X,\log}\right)$

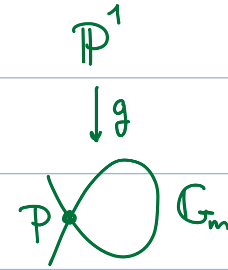
<u>Spectral sequence</u>:

$$\begin{array}{|lcl}
R^1\pi_*\mathcal{O}_{\bar{\mathcal{E}}} & \longrightarrow & R^1\pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log} \\
\pi_*\mathcal{O}_{\bar{\mathcal{E}}} & \longrightarrow & \pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log} \\ \hline
\end{array}$$

(Clear on $Y_1(N) = X_1(N) \setminus$ cusps. At the cusp,

$$\mathcal{O}_{\bar{\mathcal{E}}_x} = \ker\left(g_*\mathcal{O}_{\mathbb{P}^1_x} \twoheadrightarrow k_x\right)$$

$$f \longmapsto f(0) - f(\infty)$$



To compute $R^\bullet\pi_*\mathcal{O}_{\bar{\mathcal{E}}_x}$, we see
$$\begin{array}{|lcl}
H^1(\mathbb{P}^1_x, \mathcal{O})^{=0} & & \\
H^0(\mathbb{P}^1_x, \mathcal{O}) & \longrightarrow & H^0(k_x) \\ \hline
\text{Const function} & \longrightarrow & 0
\end{array}$$

So $R^{0,1}\pi_*\mathcal{O}_{\bar{\mathcal{E}}_x} = 1\text{-dim}'l \Rightarrow \pi_*\mathcal{O}_{\bar{\mathcal{E}}}$ and $R^1\pi_*\mathcal{O}_{\bar{\mathcal{E}}}$ are loc. free of rk 1.

$$\Omega^1_{\bar{\mathcal{E}}/X,\log,x} = \mathcal{O}_{\bar{\mathcal{E}}_x}\cdot\frac{dz}{z} = \ker\left(g_*\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\}) \twoheadrightarrow k_x\right)$$

$R^\bullet\pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log,x}$
$$\begin{array}{|lcl}
H^1(\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})) = 0 & & \\
H^0(\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})) & \longrightarrow & k_x \\ \hline
\end{array}$$
$\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\}) \simeq \mathcal{O}_{\mathbb{P}^1}$

same as above $\Rightarrow R^{0,1}\pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log}$ are loc. free of rk 1.

<u>Cor</u>:
$$0 \to \pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log} \to \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \to R^1\pi_*\mathcal{O}_{\bar{\mathcal{E}}} \to 0$$

Over $Y = X \setminus$ cusp, this is just
$$0 \to \pi_*\Omega^1_{\mathcal{E}/Y} \to H^1_{dR}(\mathcal{E}/Y) \to R^1\pi_*\mathcal{O}_{\mathcal{E}} \to 0$$
the family version of Hodge filtration.

<u>Remark</u>: Since $\Omega^1_{\mathcal{E}/Y} \simeq \mathcal{O}_{\mathcal{E}}$ is the trivial sheaf,
$$\pi_*\Omega^1_{\mathcal{E}/Y} \cong i^*\Omega^1_{\mathcal{E}/Y}.$$ So may replace $\pi_*\Omega^1_{\mathcal{E}/Y}$ by $\omega_{\mathcal{E}/Y}$.

At the cusp, $\pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log,x}$ is also rank 1, coming from $H^0(\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})) \simeq k_x$

So can also be identified with $i^*\Omega^1_{\bar{\mathcal{E}}/X,\log,x} = i^*\Omega^1_{\mathcal{E}^{sm}/X,x}$

Cor: $0 \to \omega_X \to \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \to R^1\pi_* \mathcal{O}_{\bar{\mathcal{E}}} \to 0.$

Fact: $\wedge^2 \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \simeq \mathcal{H}^2_{dR,\log}(\bar{\mathcal{E}}/X) = R^1\pi_*\Omega^1_{\bar{\mathcal{E}}/X,\log} \simeq \mathcal{O}_X$ ← relative Serre duality

Cor: $R^1\pi_*\mathcal{O}_{\bar{\mathcal{E}}} \cong \omega_X^{-1}$.

Gauss-Manin connection: "Try to compute de Rham cohomology of $\bar{\mathcal{E}}$"

$$\begin{tikzcd}
 & \pi^*\Omega^1_{X,\log} \arrow[r] \arrow[d] & \pi^*\Omega^1_{X,\log} \otimes \Omega^1_{\bar{\mathcal{E}}/X,\log} \arrow[d, equal] \\
DR_{\bar{\mathcal{E}}} := \mathcal{O}_{\bar{\mathcal{E}}} \arrow[r, "d"] \arrow[d, equal] & \Omega^1_{\bar{\mathcal{E}},\log} \arrow[r, "d"] \arrow[d] & \Omega^2_{\bar{\mathcal{E}},\log} \\
DR_{\bar{\mathcal{E}}/X,\log} := \mathcal{O}_{\bar{\mathcal{E}}} \arrow[r, "d"] & \Omega^1_{\bar{\mathcal{E}}/X,\log} &
\end{tikzcd}$$

So get an exact triangle:

$$\underbrace{R\pi_*\left(\pi^*\Omega^1_{X,\log} \to \pi^*\Omega^1_{X,\log}\otimes\Omega^1_{\bar{\mathcal{E}}/X,\log}\right)}_{\cong\ \mathcal{DR}_{\bar{\mathcal{E}}/X,\log} \otimes \Omega^1_{X,\log}[-1]}[-1] \to DR_{\bar{\mathcal{E}}} \to \mathcal{DR}_{\bar{\mathcal{E}}/X,\log} \xrightarrow{+1}$$

$\rightsquigarrow \nabla_{GM}: \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \to \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \otimes \Omega^1_{X,\log}$

<u>Gauss-Manin connection</u>.

& $H^*_{dR,\log}(\bar{\mathcal{E}}/\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}]) \cong \mathbb{H}^*(\mathcal{H}^*_{dR,\log}(\bar{\mathcal{E}}/X) \otimes \Omega^\bullet_{X,\log}, \nabla_{GM})$

deRham cohom of family can be computed in steps.

Cor:

$$\begin{tikzcd}
\omega_X \subseteq \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \arrow[r, "\nabla_{GM}"] \arrow[rd] & \mathcal{H}^1_{dR,\log}(\bar{\mathcal{E}}/X) \otimes \Omega^1_{X,\log} \arrow[d] \\
 & \omega_X^{-1} \otimes \Omega^1_{X,\log}
\end{tikzcd}$$

This is linear!

$\Rightarrow \omega_X^{\otimes 2} \to \Omega^1_{X,\log}$ Kodaira-Spencer isomorphism.