Special values of L-functions 18

——— Periods of algebraic Hecke characters II.

Corrections from last time.

- If $\chi : L^\times\backslash\mathbb{A}_L^\times \to \mathbb{C}^\times$ is an algebraic Hecke character of weight $\mu = N_{L/K}(\beta-\alpha)$

  for $K =$ max'l CM subfield of L,

  $\exists$ CM type Φ of K, $\alpha \in \mathbb{Z}_{>0}[\Phi]$, $\beta \in \mathbb{Z}_{\geq 0}[c\Phi]$

  then $$\left(\frac{(2\pi i)^{|\beta_0|}}{\Omega^{\alpha_0}\,{}^{\vee}\Omega^{\beta_0}}\right)^{[L:K]} \cdot L(\chi,0) \in \overline{\mathbb{Q}}^{\mathrm{alg},\times}$$

- Geometry of Kuga-Sato variety at cusps. $N = p_1 \cdots p_r$ say square free

  $Y(\mathbb{C}) = \Gamma\backslash\mathfrak{h}$, cusps $= \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ "locally, look at $\Gamma_p\backslash GL_2(\mathbb{F}_p)/B(\mathbb{F}_p)$"

  Each prime works "independently"

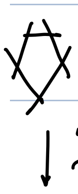
The "universal generalized elliptic curve at cusps"

$$X(\Gamma(p)) \longrightarrow X(\Gamma_1(p)) \longrightarrow X(\Gamma_0(p)) \longrightarrow \text{"}X(1)\text{"}$$

$X(\Gamma_1(p))$: classify embedding $\mu_p \hookrightarrow E[p]$; $X(\Gamma_0(p))$: classifies subgroup of order p

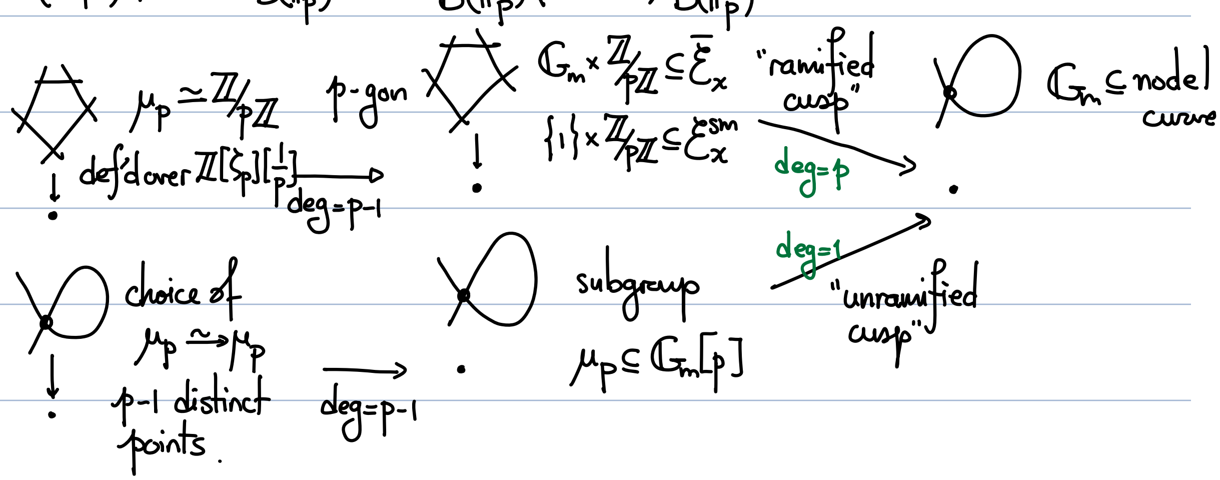
$$\begin{pmatrix}1 & \mathbb{F}_p\\ & 1\end{pmatrix}\backslash GL_2(\mathbb{F}_p)/B(\mathbb{F}_p) \qquad B(\mathbb{F}_p)\backslash GL_2(\mathbb{F}_p)/B(\mathbb{F}_p)$$

Every cusp point is a p-gon



$\mu_p \times \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mathbb{G}_m[p] \times \mathbb{Z}/p\mathbb{Z}$.

But the isom need not to be defined over $\mathbb{Z}[\frac{1}{p}]$



Remark: This does not affect the log-geometry computation.

§1. Gauss-Manin connection

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}^{sm} \subseteq \overline{\mathcal{E}} \arrow[d, "\pi"] & \supseteq \mathcal{D} \arrow[d] \\ Y_1(N)=Y \arrow[r] & X_1(N)=X \arrow[u, bend left, "\iota"] & C \arrow[l] \end{tikzcd}$$

$$0 \to \pi^*\Omega^1_{X_1(N)/\mathbb{Z}[\frac{1}{N}]}(\log C) \to \Omega^1_{\overline{\mathcal{E}}/\mathbb{Z}[\frac{1}{N}]}(\log D) \to \Omega^1_{(\overline{\mathcal{E}},D)/(X_1(N),C)} \to 0$$

$$\Omega^1_{X,\log} \qquad \Omega^1_{\overline{\mathcal{E}},\log} \qquad \Omega^1_{\overline{\mathcal{E}}/X,\log}$$

$$\omega := \iota^*\Omega^1_{\mathcal{E}^{sm}/X} \cong \pi_*\Omega^1_{\overline{\mathcal{E}}/X,\log}$$

$$0 \to \omega \to R^1\pi_*\left(\mathcal{O}_{\overline{\mathcal{E}}} \to \Omega^1_{\overline{\mathcal{E}}/X,\log}\right) \to \omega^{-1} \to 0$$

$$R^1\pi_*\left(\mathcal{O}_{\overline{\mathcal{E}}} \to \Omega^1_{\overline{\mathcal{E}}/X,\log}\right) = \mathcal{H}^1_{dR,\log}(\overline{\mathcal{E}}/X)$$

b/c $\Lambda^2 \mathcal{H}^1_{dR,\log}(\overline{\mathcal{E}}/X) \xrightarrow{\sim} R^2\pi_*\left(\mathcal{O}_{\overline{\mathcal{E}}} \to \Omega^1_{\overline{\mathcal{E}}/X,\log}\right) \simeq \mathcal{O}_{\overline{\mathcal{E}}}$.

Gauss-Manin connection: "Try to compute de Rham cohomology of $\overline{\mathcal{E}}$"

$$\begin{tikzcd} & \pi^*\Omega^1_{X,\log} \arrow[r] \arrow[d] & \pi^*\Omega^1_{X,\log}\otimes\Omega^1_{\overline{\mathcal{E}}/X,\log} \arrow[d, equal] \\ DR_{\overline{\mathcal{E}}} := \mathcal{O}_{\overline{\mathcal{E}}} \arrow[r, "d"] \arrow[d, equal] & \Omega^1_{\overline{\mathcal{E}},\log} \arrow[r, "d"] \arrow[d] & \Omega^2_{\overline{\mathcal{E}},\log} \\ DR_{\overline{\mathcal{E}}/X,\log} := \mathcal{O}_{\overline{\mathcal{E}}} \arrow[r, "d"] & \Omega^1_{\overline{\mathcal{E}}/X,\log} & \end{tikzcd}$$

So get an exact triangle:

$$\underbrace{R\pi_*\left(\pi^*\Omega^1_{X,\log} \to \pi^*\Omega^1_{X,\log}\otimes\Omega^1_{\overline{\mathcal{E}}/X,\log}\right)}_{\cong \text{ projection formula}}[-1] \to DR_{\overline{\mathcal{E}}} \to DR_{\overline{\mathcal{E}}/X,\log} \xrightarrow{+1}$$

$$DR_{\overline{\mathcal{E}}/X,\log}\otimes\Omega^1_{X,\log}[-1]$$

$$\rightsquigarrow \nabla_{GM}: \mathcal{H}^1_{dR,\log}(\overline{\mathcal{E}}/X) \to \mathcal{H}^1_{dR,\log}(\overline{\mathcal{E}}/X)\otimes\Omega^1_{X,\log}$$

<u>Gauss-Manin connection</u>.

& $\mathcal{H}^*_{dR,\log}(\overline{\mathcal{E}}/\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]) \cong \mathbb{H}^*\left(\mathcal{H}^*_{dR,\log}(\overline{\mathcal{E}}/X)\otimes\Omega^\bullet_{X,\log}, \nabla_{GM}\right)$

i.e. deRham cohom of family can be computed in steps.

Relation to "Riemann-Hilbert correspondence" (ignore the log-pole)

X smooth projective variety $/\mathbb{C}$. There's an equivalence of categories

$$\left\{\begin{matrix} \underline{\mathbb{C}}\text{-local system on } X(\mathbb{C})^{an} \\ \leftrightarrow \text{representation } \pi_1(X(\mathbb{C})^{an},x)\to GL_N(\mathbb{C}) \end{matrix}\right\} \overset{R\text{-}H}{\longleftrightarrow} \left\{\begin{matrix} \text{rank } n \text{ vector bundle } \mathcal{E}/X \text{ together} \\ \text{with integrable connection } \nabla: \mathcal{E}\to\mathcal{E}\otimes\Omega^1_X \end{matrix}\right\}$$

means $\mathcal{E} \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^1_X \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^2_X$, composite 0

$$\underline{\mathbb{L}} \longmapsto (\underline{\mathbb{L}}\otimes_{\underline{\mathbb{C}}}\mathcal{O}_X,\ 1\otimes d)$$

Solution sheaf $\underline{Sol}(\mathcal{E},\nabla) \longleftarrow (\mathcal{E},\nabla)$

locally $U\subseteq X(\mathbb{C})^{an}$, $\underline{Sol}(\mathcal{E},\nabla)(U):=\Gamma(U,\mathcal{E})^{\nabla=0}$

space of horizontal sections

Katz-Oda: Let $X \xrightarrow{\pi} S/\mathbb{C}$ be a proper smooth morphism with S smooth$/\mathbb{C}$.

Let $\underline{\mathbb{L}}$ be a $\underline{\mathbb{C}}$-local system on $X \overset{RH}{\longleftrightarrow} (M,\nabla)$ vector bundle with integrable connections

We may consider the complex $M\otimes\Omega^\bullet_{X/S} := [M \xrightarrow{\nabla} M\otimes\Omega^1_{X/S} \xrightarrow{\nabla} M\otimes\Omega^2_{X/S} \xrightarrow{\nabla} \cdots]$

$\pi^i_{dR}(M,\nabla) := R^i\pi_*(M\otimes\Omega^\bullet_{X/S})$ carries a Gauss-Manin connection

Then $\forall i$, $R^i\pi_*\underline{\mathbb{L}}$ is a $\underline{\mathbb{C}}$-local system on $S \overset{R\text{-}H}{\longleftrightarrow} (\pi^i_{dR}(M,\nabla),\nabla_{GM})$

in the sense that $(R^i\pi_*\underline{\mathbb{L}}\otimes\mathcal{O}_{S^{an}},1\otimes d)\cong(\pi^i_{dR}(M,\nabla),\nabla_{GM})$

In our case, $\mathcal{E} \xrightarrow{\pi} Y$: $R^1\pi_*\underline{\mathbb{Z}}_{\mathcal{E}^{an}} =: \mathcal{H}^1_B(\mathcal{E}/Y)$ is a rank 2 $\underline{\mathbb{Z}}$-local system $/Y$

We have $(\mathcal{H}^1_B(\mathcal{E}/Y)\otimes_{\underline{\mathbb{Z}}}\mathcal{O}_{Y^{an}},1\otimes d)\cong(\mathcal{H}^1_{dR}(\mathcal{E}/Y)^{an},\nabla_{GM})$

§2 Kodaira-Spencer isomorphism

$$\begin{tikzcd}
\omega_{\bar{\mathcal E}} \subseteq \mathcal H^1_{dR,\log}(\bar{\mathcal E}/X) \arrow[r, "\nabla_{GM}"] \arrow[dr] & \mathcal H^1_{dR,\log}(\bar{\mathcal E}/X)\otimes\Omega^1_{X,\log} \arrow[d] \\
& \omega_{\bar{\mathcal E}}^{-1}\otimes\Omega^1_{X,\log}
\end{tikzcd}$$

This is $\mathcal O_X$-linear!

(b/c for $a\in\mathcal O_X$, $x\in\omega_{\bar{\mathcal E}}$, $\nabla_{GM}(a\cdot x) = \underbrace{x\otimes da}_{\text{belongs to } \omega_{\bar{\mathcal E}}\otimes\Omega^1_{X,\log}} + a\cdot\nabla_{GM}(x)$)

$\Rightarrow$ $\omega_{\bar{\mathcal E}}^{\otimes 2}\longrightarrow\Omega^1_{X,\log}$ Kodaira-Spencer map.

__Theorem__. $\omega_{\bar{\mathcal E}}^{\otimes 2}\xrightarrow{\sim}\Omega^1_{X,\log}$ is an isomorphism over $X/\mathbb Z[\frac{1}{N}]$.

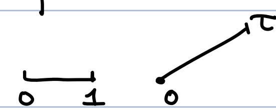
Proof: Will only prove this over $\mathbb Q$ (or equivalently $\mathbb C$), & over Y.

for the integral version, need deformation theory of abelian varieties and p-divisible groups.

* __Local computation of Gauss-Manin connection__:

$$\begin{tikzcd}
\mathbb C_z/\mathbb Z\oplus\mathbb Z\tau \subseteq \mathcal E_U^{an} \arrow[d] & \\
\tau \in U^{an} \subseteq \mathfrak h \arrow[dr] & \\
& \Gamma\backslash\mathfrak h
\end{tikzcd}$$

Work locally, $\mathcal H^1_B(\mathcal E_U^{an}/U^{an})$ is free of rank 2

with basis X, Y, dual to 

$\omega_{\bar{\mathcal E}}|_U$ is generated by dz for z the parameter on $\mathbb C/\mathbb Z\oplus\mathbb Z\tau$.

Under the isomorphism $\mathcal H^1_{dR}(\mathcal E_U^{an}/U^{an}) \simeq \mathcal H^1_B(\mathcal E_U^{an}/U^{an})\otimes_{\mathbb Z}\mathcal O_U^{an}$

$$\begin{array}{c} \mathcal H^1_{dR}(\mathcal E_U^{an}/U^{an}) \\ \cup\!\mid \\ dz\in\omega_{\bar{\mathcal E}} \end{array}$$

we have $dz = \left(\int_0^1 dz\right)\cdot X + \left(\int_0^\tau dz\right)Y = X+\tau Y$.

We also need to interpret $\mathcal H^1_{dR}(\mathcal E_U^{an}/U^{an}) \twoheadrightarrow \underset{\sim}{\omega_{\bar{\mathcal E}}^{-1}}$

$\omega_{\bar{\mathcal E}}^{-1} \cong \mathrm{Hom}(\omega, \wedge^2\mathcal H^1_{dR}(\mathcal E_U^{an}/U^{an}))$

$\eta \longmapsto$ "cupping with η" i.e. $dz\mapsto dz\wedge\eta$

__Note__: $\nabla_{GM}(X)=\nabla_{GM}(Y)=0$. $\Rightarrow \nabla_{GM}(dz)=\nabla_{GM}(X+\tau Y) = Y\otimes d\tau$

Its projection to $\omega_{\bar{\mathcal E}}^{-1}$ is $dz\mapsto\langle X+\tau Y, Y\rangle\otimes d\tau =$ "1"$\otimes d\tau$ $\qquad\langle X,Y\rangle=$ "1"

So its projection to $\omega_{\bar{\mathcal E}}^{-1}$ is $(dz)^*\otimes d\tau$

Thus, the induced map $\omega_{\bar{\mathcal E}}\xrightarrow{\text{pro}\,\nabla_{GM}}\omega_{\bar{\mathcal E}}^{-1}$ is $dz\mapsto(dz)^*\otimes d\tau$. ✓

§3 Splitting of Hodge filtration.

Katz observed that the special values of L-function associated to algebraic Hecke character is related to the splitting of Hodge filtration.

* $\omega_{\mathcal{E}} \subseteq \mathcal{H}^1_{dR}(\mathcal{E}/Y)$.

Write $A_Y :=$ sheaf of C^∞-functions on $Y(\mathbb{C})$

<u>Fact</u>: $\mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes_{\mathcal{O}_Y} A_Y \cong \omega_{\mathcal{E}}\otimes_{\mathcal{O}_Y} A_Y \oplus \overline{\omega_{\mathcal{E}}}\otimes_{\overline{\mathcal{O}}_Y} A_Y$. $\quad (*)$

local basis: $X, Y \qquad X+\tau Y \qquad X+\bar{\tau} Y$.

At each point $y\in Y$, $(*)$ gives the canonical Hodge decomposition.

• Rewrite Gauss-Manin connection:

projection along $\overline{\omega_{\mathcal{E}}}$

$$\omega_{\mathcal{E}} \xrightarrow{\nabla_{GM}} \mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes\Omega^1_Y \subseteq \mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes\Omega^1_Y\otimes_{\mathcal{O}_Y} A_Y \overset{pr}{\twoheadrightarrow} \omega_{\mathcal{E}}\otimes\Omega^1_Y\otimes_{\mathcal{O}_Y} A_Y \overset{K\text{-}S}{\cong} \omega_{\mathcal{E}}^{3}\otimes A_Y$$

$$\underset{\substack{\| \\ dz}}{f(\tau)\otimes(X+\tau Y)} \longmapsto \left(\frac{df}{d\tau}\otimes(X+\tau Y)+f(\tau)\otimes Y\right)\otimes d\tau \longmapsto \left(\frac{df}{d\tau}+\frac{f}{2iv}\right)\otimes(X+\tau Y)\otimes d\tau$$

write $Y=\frac{1}{\tau-\bar\tau}\big((X+\tau Y)-(X+\bar\tau Y)\big)$ $\qquad$ write $\tau=u+iv$

$$\simeq \left(\frac{d}{d\tau}+\frac{1}{2iv}\right)(f)\otimes dz^{\otimes 3}$$

More generally, $D_k: \mathrm{Sym}^k\mathcal{H}^1_{dR}(\mathcal{E}/Y) \xrightarrow{\nabla_{GM}} \mathrm{Sym}^k\mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes\Omega^1_Y$

$$\overset{K\text{-}S}{\cong} \mathrm{Sym}^k\mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes\omega_{\mathcal{E}}^2$$

$$\hookrightarrow \mathrm{Sym}^k\mathcal{H}^1_{dR}(\mathcal{E}/Y)\otimes\mathrm{Sym}^2\mathcal{H}^1_{dR}(\mathcal{E}/Y) \xrightarrow{prod} \mathrm{Sym}^{k+2}\mathcal{H}^1_{dR}(\mathcal{E}/Y)$$

<u>Key point</u>: $\omega^k \subseteq \mathrm{Sym}^k\mathcal{H}^1_{dR}(\mathcal{E}/Y) \xrightarrow{D_k} \mathrm{Sym}^{k+2}\mathcal{H}^1_{dR}(\mathcal{E}/Y) \xrightarrow{D_{k+2}} \mathrm{Sym}^{k+4}\mathcal{H}^1_{dR}(\mathcal{E}/Y)$

$\cup$

Eisenstein series E_k

$$\to \cdots \to \mathrm{Sym}^{k+2r}\mathcal{H}^1_{dR}(\mathcal{E}/Y)$$

$$\downarrow \text{non-holomorphic projection}$$

$$\omega^{k+2r}\otimes A_Y$$

$\to$ non-holomorphic Eisenstein series $E_{k,r}$ s.t. $E_{k,r}(\tau_0) = (*)\, L(\chi, 0)$ $\to$ for χ algebraic Hecke char of $\mathbb{Q}(\sqrt{-D})$, $\tau\approx\sqrt{-D}$.

On the other hand, E_k itself is algebraic, and $\mathbb{D}_{k+2r-2} \circ \cdots \circ \mathbb{D}_k(E_k) \in \Gamma(Y, \mathrm{Sym}^{k+2r} \mathcal{H}^1_{dR}(\mathcal{E}/Y))$

is also algebraic.

The "non-algebraic operation" is the non-holomorphic projection

$$
\begin{tikzcd}
 & \omega^{k+2r} \otimes A_Y \arrow[dr, "\mathrm{eval}_{\tau_0}"] & \\
\mathrm{Sym}^{k+2r} \mathcal{H}^1_{dR}(\mathcal{E}/Y) \arrow[ur, "\text{non-holom. proj.}"] \arrow[dr, "\mathrm{eval}_{\tau_0}"'] & & H^0(E_{\tau_0}, \Omega^1) \otimes \mathbb{C} \\
 & \underbrace{\mathrm{Sym}^{k+2r}_{k(P_{\tau_0})} H^1_{dR}(E_{\tau_0})} \arrow[ur, "\text{projection}"'] &
\end{tikzcd}
$$

if $\tau_0 \in \mathbb{Q}(\sqrt{-D})$, $\underbrace{H^1_{dR}(E_{\tau_0})} \circlearrowleft \mathbb{Q}(\sqrt{-D})$

free $\mathbb{Q}(\sqrt{-D}) \otimes k(P_{\tau_0})$-module of rank 1

But this projection is to separate $\mathbb{Q}(\sqrt{-D})$-eigenspaces,

so the operation is algebraic!

<u>Formula</u>: $\mathbb{D}_k : \mathrm{Sym}^k \mathcal{H}^1_{dR}(\mathcal{E}/Y) \otimes A_Y \xrightarrow{\nabla_{GM}} \mathrm{Sym}^{k+2} \mathcal{H}^1_{dR}(\mathcal{E}/Y) \otimes A_Y$

$$\nabla_{GM}\left(f(\tau) \otimes (X+\tau Y)^{\otimes i} \otimes (X+\bar\tau Y)^{\otimes(k-i)}\right)$$

$$= \frac{\partial}{\partial \tau}(f) \otimes d\tau \otimes (X+\tau Y)^{\otimes i} \otimes (X+\bar\tau Y)^{\otimes(k-i)} + f \otimes i\, Y d\tau \otimes (X+\tau Y)^{i-1} \otimes (X+\bar\tau Y)^{k-i}$$

$$= \frac{\partial}{\partial \tau}(f) \otimes (X+\tau Y)^{\otimes(i+2)} \otimes (X+\bar\tau Y)^{\otimes(k-i)} + \frac{i}{\tau-\bar\tau} f \cdot \left((X+\tau Y) - (X+\bar\tau Y)\right) \otimes (X+\tau Y)^{i+1} \otimes (X+\bar\tau Y)^{k-i}$$

$$= \left(\frac{\partial}{\partial \tau}(f) + \frac{i}{\tau-\bar\tau} f\right) \otimes (X+\tau Y)^{\otimes(i+2)} \otimes (X+\bar\tau Y)^{\otimes(k-i)} - \frac{i}{\tau-\bar\tau} f \otimes (X+\tau Y)^{i+1} \otimes (X+\bar\tau Y)^{k-i+1}.$$

In particular, if $f \otimes dz^k \in H^0(Y, \omega^k) \subseteq H^0(Y, \mathrm{Sym}^k(\mathcal{E}/Y))$, for $r \geq 2$

$$\mathrm{pr}_{k+2r}\left(\mathbb{D}_{k+2r-2} \circ \cdots \circ \mathbb{D}_{k-2} \circ \mathbb{D}_k(f \otimes dz^k)\right)$$

$$= \left(\frac{\partial}{\partial \tau} + \frac{k+2r-2}{\tau-\bar\tau}\right) \circ \cdots \circ \left(\frac{\partial}{\partial \tau} + \frac{k+2}{\tau-\bar\tau}\right) \circ \left(\frac{\partial}{\partial \tau} + \frac{k}{\tau-\bar\tau}\right)(f) \otimes dz^{k+2r}.$$