Special values of L-functions 20

—— Introduction to Beilinson's conjecture

§1. Beilinson's conjecture (rough form)

Let X be a smooth projective variety $/\mathbb{Q}$

We will be interested in $M = H^i(X, \mathbb{Q}(n))$ of weight $w = i-2n$.

$$\rightsquigarrow L(M,s) = L(H^i(X,\mathbb{Q}(n)),s) = L(H^i(X,\mathbb{Q}), n+s)$$

$$= \prod_{\substack{p \text{ prime} \\ \text{good}}} \frac{1}{\det\left(id - Fr_p \cdot p^{-n-s}; H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_{\mathbb{Q}_p}}\right)}$$

Proposition $L(M,0)$ is critical $\iff$ $\alpha_M : \underbrace{M_B^+}_{F_\infty \text{ acts trivially}} \otimes \mathbb{C} \subseteq \underbrace{M_B \otimes \mathbb{C}}_{\circlearrowleft\, F_\infty \otimes c} \cong M_{dR} \otimes \mathbb{C} \twoheadrightarrow M_{dR}/F^0 M_{dR} \otimes \overset{\curvearrowright\, c}{\mathbb{C}}$ is an isomorphism

Beilinson's conjecture: What if α_M is not an isomorphism?

* **Recall**: $L_\infty(M,s) = \prod_{p \neq q} \Gamma_{\mathbb{C}}(s - \min\{p,q\})^{\dim H^{pq}} \cdot \prod_{p=q} \Gamma_{\mathbb{R}}(s-p)^{\dim H^{pp, F_\infty = (-1)^p}} \cdot \Gamma_{\mathbb{R}}(s-p+1)^{\dim H^{pp, F_\infty = (-1)^{p+1}}}$

$\underline{w \geq 1} \Leftarrow$ has a pole at $s=0$ iff $\min\{p,q\} \geq 0$
$\Updownarrow$
H^{pq} and H^{qp} both in $F^0 M_{dR}$
$\Updownarrow$
$\alpha_{pq} : (H^{pq} \oplus H^{qp})^{F_\infty = 1} \longrightarrow (H^{pq} \oplus H^{qp})/F^0$ has kernel $= \dim H^{pq}$

has a pole at $s=0$ if $p \geq 0$ and p even / p odd
$\Updownarrow$ $\Rightarrow$ $\underline{w \geq 0}$

If $p \geq 0$, $\underbrace{H^{pp, F_\infty = 1} \longrightarrow H^{pp}/F^0 = 0}_{\text{kernel dim } + 1}$

Conclusion: $\operatorname{ord}_{s=0} L_\infty(M,s) = \dim \operatorname{Ker} \alpha_M$ $\neq 0 \Rightarrow w(M) \geq 0$

$\operatorname{ord}_{s=0} L_\infty(M^\vee(1), s) = \dim \operatorname{Ker} \alpha_{M^\vee(1)} = \dim \operatorname{Coker} \alpha_M$. $\neq 0 \Rightarrow w(M) \leq -2$

(Conjectural) **functional equation**: (if $M = H^i(X)(n)$ is self dual, $M = M^\vee(-w)$)

$$L(M,s) \cdot L_\infty(M,s) = \varepsilon(M,s) \cdot L(M^\vee(1), -s)\, L_\infty(M^\vee(1), -s)$$

$$\overset{\text{selfdual}}{=} \varepsilon(M,s) \cdot L(M, w+1-s) \cdot L_\infty(M, w+1-s) \quad \text{center Re } s = \frac{w+1}{2}$$

<u>Beilinson's conjecture</u>. (When $M = H^i(X,\mathbb{Q})(n)$), assume $w(M) \leq -2$ and $\mathrm{Coker}\,\alpha_M \neq 0$

(Assume moreover M doesn't contain $\mathbb{Q}(1)$ as a direct factor)

Then (1) $L(M,0) \neq 0$ (in fact, this is expected to be "provable" if $w(M) \leq -3$ by Deligne's weight theory + Weight monodromy conjecture. b/c $\prod_p L_p(M,0)$ converges.)

expected $\mathrm{ord}_{s=0} L(M^\vee(1),0) = \dim \mathrm{Coker}\,\alpha_M$.

(2) There exist some "motivic group"

$$\mathrm{reg}_\infty : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n)) \longrightarrow \mathrm{Coker}\left(H^i(X,\mathbb{C}(n)) \xrightarrow{\alpha_M} H^i_{dR}(X/\mathbb{C})/F^n H^i_{dR}(X/\mathbb{C})\right)$$

(note $w = i - 2n \leq -2 \Rightarrow n \geq \frac{i}{2}+1$; so $\dim F^n$ is $\underset{\text{typically}}{<} \frac{1}{2}\dim H^i_{dR}$)

s.t. reg_∞ is an isomorphism.

Moreover, $\mathrm{reg}_\infty\left(\det H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))\right) = L(M,0)\cdot\det\left(\underset{\sim}{\mathrm{Coker}\,\alpha_M}\right)$ as rational structures

Or equivalently, $\det\left(\dfrac{H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))}{H^i(X,\mathbb{Q}(n))} \otimes \mathbb{R} \xrightarrow{\widetilde{\mathrm{reg}}_\infty} \dfrac{H^i_{dR}(X/\mathbb{Q})}{F^n H^i_{dR}(X/\mathbb{Q})} \otimes \mathbb{R}\right) \underset{\mathbb{Q}^\times}{\sim} L(M,0).$

(need to extend $H^{i+1}_{\mathcal{M}}$)

<u>Examples</u> ① $M = \mathbb{Q}(\chi)$ for $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a Dirichlet character

$\chi(-1) = 1$. Take n odd. $\alpha_{M(n)} : M(n)^+_B = 0 \longrightarrow M_{dR}/F^n M_{dR} \otimes \mathbb{R} = M_{dR} \otimes \mathbb{R}$

$\mathrm{Coker}(\alpha_{M(n)}) = 0$, expect some motivic class to map to M_{dR}.

<u>Variant</u>: $F/\mathbb{Q}$ number field, $M = (\mathrm{Spec}\,F)(1)$ with $\mathbb{Q}(1)$ removed $\leftrightarrow \mathbb{Q}[\mathrm{Hom}(F,\mathbb{C})]/\mathbb{Q}$

$$\alpha_M : \underbrace{M^+_B \otimes \mathbb{R}}_{\text{rank} = r_2} \longrightarrow M_{dR}/F^1 M_{dR} \otimes \mathbb{R} = \underbrace{F \otimes \mathbb{R}/\mathbb{R}}_{\text{rank} = r_1 + 2r_2 - 1}$$

E.g. $n=1$, $\mathrm{reg} : \underbrace{\mathcal{O}_F^\times}_{\text{rank} = r_1 + r_2 - 1} \longrightarrow \mathrm{Coker}\,\alpha_M = \left(\prod_{\substack{\tau_i : F \to \mathbb{R} \\ \sigma_i, \sigma_i^c : F \to \mathbb{C}}} \mathbb{R}\right)^{\mathrm{Tr}=0}$

Beilinson conjecture says $\left(\zeta_F(s)/\zeta_{\mathbb{Q}}(s)\right)\Big|_{s=1} \in \mathrm{Reg}_F \cdot (2\pi i)^{r_2} \cdot \overline{\mathbb{Q}}^\times$

② Borel's higher regulators for $F/\mathbb{Q}$ with larger n (later)

③ <u>Beilinson</u> for modular form f of wt 2. $L(f,2)$ (center $=1$) also higher wt version

"K_2 of modular curve"

for two modular forms f, g of wt 2 $L(f \times g, 2)$ (center $=\frac{3}{2}$). ——— ✓

$$\theta \text{ Sigel element} \quad X \overset{\Delta}{\hookrightarrow} X \times X$$

④ Later works $\quad \overset{\theta \text{ Siegel}}{X} \hookrightarrow S$ — Hilber modular surface / $U(1,2)$ Shimura variety

$$\overset{\theta}{X} \times \overset{\theta}{X} \hookrightarrow Sh_{GSp_4}$$

& more small dimensional cases. No ≥ 10 dim nontriv cases yet.

§2 <u>Deligne's cohomology</u>.

* Instead of working with $\mathrm{Coker}\left(\alpha_M : H^i_B(X, \mathbb{R}(n)) \to H^i_{dR}(X/\mathbb{C})/F^n\right)$,

we do this at the sheaf level.

Let $X = X^{an}$ be a smooth projective analytic variety $/\mathbb{C}$, the <u>Deligne complex</u> is the following

For $A \subseteq \mathbb{R}$ a subring, $\underline{A}_{\mathcal{D}}(n) := \left[\underline{A}(n) \to \mathcal{O}_{X^{an}} \to \Omega^1_{X^{an}} \to \cdots \to \Omega^{n-1}_{X^{an}}\right]$ ⟲ If X is defined over $\mathbb{R}$, get an $F_\infty \otimes c$-action

(n) means $(2\pi i)^n$

$$= \mathrm{Cone}\left[\begin{matrix} A(n) \\ \oplus \\ \underbrace{\boxed{\Omega^{\geq n}_{X^{an}}}}_{= F^n \Omega^{\bullet}_{X^{an}}} \end{matrix} \longrightarrow \Omega^{\bullet}_{X^{an}}\right][-1]$$

By definition, cohomology of $\underline{A}_{\mathcal{D}}(n)$ fits in the following exact sequence:

$$H^i_B(X, A(n)) \to \underset{\substack{\text{IIS} \\ H^i_{dR}(X^{an}) / F^n H^i_{dR}(X^{an})}}{H^i_{dR}\left(X^{an}, \mathcal{O}_{X^{an}} \to \cdots \to \Omega^{n-1}_{X^{an}}\right)} \to \underbrace{H^{i+1}\left(X^{an}, \underline{A}_{\mathcal{D}}(n)\right)}_{\substack{\text{interested in} \\ \text{this } F_\infty \otimes c = 1 \text{ part}}} \to \underbrace{H^{i+1}_B\left(X^{an}, A(n)\right) \to \frac{H^{i+1}_{dR}(X^{an})}{F^n H^{i+1}_{dR}}}_{\substack{\text{when } A = \mathbb{R}, \text{ this is injective.} \\ \text{when } n \geq \frac{i}{2}+1.}}$$

• <u>Properties</u>: ① $\mathbb{Z}_{\mathcal{D}}(1) = [\mathbb{Z}(1) \to \mathcal{O}_{X^{an}}] = \mathcal{O}^{\times}_{X^{an}}[1]$

In particular, $H^1(X^{an}, \mathbb{Z}_{\mathcal{D}}(1)) = H^0(X^{an}, \mathcal{O}^{\times}_{X^{an}}) = \mathcal{O}(X^{an})^{\times}$ (okay even when X is not projective)

$$H^2(X^{an}, \mathbb{Z}_{\mathcal{D}}(1)) = H^1(X^{an}, \mathcal{O}^{\times}_{X^{an}}) = \mathrm{Pic}(X^{an})$$

$$\left(\begin{tikzcd} H^2(X^{an}, \mathbb{Z}_{\mathcal{D}}(1)) \arrow[r] \arrow[d, equal] & H^2(X^{an}, \mathbb{Z}(1)) \\ \mathrm{Pic}(X^{an}) \arrow[ur, "\text{``chern class''}"'] & \end{tikzcd}\right)$$

② <u>Cup product</u>: $H^i(X^{an}, A_{\mathcal{D}}(m)) \times H^j(X^{an}, A_{\mathcal{D}}(n)) \xrightarrow{\cup} H^{i+j}(X^{an}, A_{\mathcal{D}}(m+n))$

<u>General setup</u>: Given complexes $C_1^\bullet, C_2^\bullet, D_1^\bullet, D_2^\bullet$ and

morphisms $f_1, g_1 : C_1^\bullet \to D_1^\bullet$; $f_2, g_2 : C_2^\bullet \to D_2^\bullet$

(In example, $f_i : A_{\mathcal{D}}(m) \oplus \Omega_X^{\geq m} \twoheadrightarrow A_{\mathcal{D}}(m) \hookrightarrow \Omega_X^\bullet$; $g_i : A_{\mathcal{D}}(m) \oplus \Omega_X^{\geq m} \twoheadrightarrow \Omega_X^{\geq m} \hookrightarrow \Omega_X^\bullet$

Construct $[C_1^\bullet \xrightarrow{f_1 - g_1} D_1^\bullet] \otimes [C_2^\bullet \xrightarrow{f_2 - g_2} D_2^\bullet] \xrightarrow{?} [C_1^\bullet \otimes C_2^\bullet \xrightarrow{(f_1 \otimes f_2) - (g_1 \otimes g_2)} D_1^\bullet \otimes D_2^\bullet]$

means $\mathrm{Cone}[C_1^\bullet \xrightarrow{f_1 - g_1} D_1^\bullet][-1]$

Pick any $a \in A$ (and the result is homotopic for different a).

$$\begin{tikzcd}[column sep=huge] C_1^\bullet \otimes C_2^\bullet \arrow[r, "{(f_1-g_1)\otimes 1,\ 1\otimes(f_2-g_2)}"] \arrow[d, equal, "id"] & D_1^\bullet \otimes C_2^\bullet \oplus C_1^\bullet \otimes D_2^\bullet \arrow[r, "1\otimes(g_2-f_2)\oplus(f_1-g_1)\otimes 1"] \arrow[d, "{a\cdot(1\otimes g_2 \oplus f_1\otimes 1) + (1-a)\cdot(1\otimes f_2 \oplus g_1 \otimes 1)}"] & D_1^\bullet \otimes D_2^\bullet \\ C_1^\bullet \otimes C_2^\bullet \arrow[r, "f_1\otimes f_2 - g_1 \otimes g_2"] & D_1^\bullet \otimes D_2^\bullet & \end{tikzcd}$$

$\mathrm{Cup}_a \downarrow$

check: $(x \otimes y) \longmapsto (f_1 - g_1)(x) \otimes y + x \otimes (f_2 - g_2)(y)$

$\downarrow$

$a \cdot \big((f_1 - g_1)(x) \otimes g_2(y) + f_1(x) \otimes (f_2 - g_2)(y)\big) = a \cdot \big(f_1(x) \otimes f_2(y) - g_1(x) \otimes g_2(y)\big)$

$+ (1-a) \cdot \big((f_1 - g_1)(x) \otimes f_2(y) + g_1(x) \otimes (f_2 - g_2)(y)\big) = (1-a)\big(f_1(x) \otimes f_2(y) - g_1(x) \otimes g_2(y)\big)$ ✓

check homotopic

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r, "d^1"] \arrow[d, "\mathrm{Cup}_a - \mathrm{Cup}_0"] \arrow[dl, dashed, "h_1 = 0"'] & \bullet \arrow[dl, dashed, "h_2 ?"] \\ \bullet \arrow[r] & \bullet & \end{tikzcd}$$

$\mathrm{Cup}_a - \mathrm{Cup}_0 = a \cdot 1 \otimes (g_2 - f_2) + a \cdot (f_1 - g_1) \otimes 1 = a \cdot d^1$. Take $h_2 = a$ ✓.

③ Gysin isomorphisms $Y \overset{i}{\subseteq} X$ is a regular subvariety of codimension d.

Then we have a morphism $\imath_* : H^i(Y^{an}, \underline{A}_{\mathcal{D}}(m)) \longrightarrow H^{i+2d}(X^{an}, \underline{A}_{\mathcal{D}}(m+d))$

need $\imath_* \underline{A}_{\mathcal{D},Y}(m) \longrightarrow \underline{A}_{\mathcal{D},X}(m+d)[2d]$

<u>Expectation</u> : The corresponding motivic cohomology have similar properties: $X/$

$$
\begin{tikzcd}
 & H^i_{et}(X, \mathbb{Q}_\ell(n)) \Leftarrow H^\bullet(\mathrm{Gal}_{\mathbb{Q}}, H^\bullet(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(n)) \\
H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \arrow[ur, "\text{étale realization}"] \arrow[dr, "\text{Deligne realization (regulator)}"'] & \\
 & H^i(X(\mathbb{C})^{an}, \mathbb{Z}_{\mathcal{D}}(n))^{F_\infty \otimes c = 1} \longrightarrow H^i(X(\mathbb{C})^{an}, \mathbb{R}_{\mathcal{D}}(n))^{F_\infty \otimes c = 1}
\end{tikzcd}
$$

Expectation of $H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \simeq H^i(X, \mathbb{Q}_{\mathcal{D}}(n))$

	H^0	H^1	H^2	H^3	H^4
$\mathbb{Q}(0)$	$\mathbb{Q}$	0	0	0	0
$\mathbb{Q}(1)$	0	$H^0(X, \mathcal{O}_X^\times)$	$\mathrm{Pic}(X) \otimes \mathbb{Q}$	0	0
$\mathbb{Q}(2)$	...	...	higher relations	relations for CH^2	$CH^2(X) \otimes \mathbb{Q}$

§3 <u>Bloch's higher Chow group</u> (one version of $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$)

Let X be a variety over a field k.

For $p > 0$, we have
$$CH^p(X) := \frac{\mathbb{Z}\langle \text{codim } p \text{ cycles on } X \rangle}{\langle \mathrm{div}(f) \mid f \text{ meromorphic function on codim } p-1 \text{ of } X \rangle}$$

<u>Bloch's interpretation</u>: $C^p(X) := \mathbb{Z}\langle \text{codim } p \text{ cycles on } X \rangle$

$C^p(X \times \mathbb{A}^1)' := \mathbb{Z}\langle \text{codim } p \text{ cycles on } X \times \mathbb{A}^1 \text{ that intersect } X \times \{0\} \text{ and } X \times \{1\} \text{ properly} \rangle$

Then $CH^p(X) \cong \mathrm{Coker}(C^p(X \times \mathbb{A}^1)' \rightrightarrows C^p(X))$

$$Z \mapsto Z \cap (X \times \{0\}), \qquad Z \mapsto Z \cap (X \times \{1\})$$

Here, the map $Z \to \mathbb{A}^1$ gives a function on $Z \rightsquigarrow$ identify $(\mathbb{A}^1, 0, 1)$ with $(\mathbb{P}^1 \setminus \{1\}, 0, \infty)$

Let $\Delta^n := \{(x_0, x_1, \cdots, x_n) \,;\, \sum x_i = 1\}$ $\quad \mathbb{A}^1 = \Delta^1 = \{(x_0, x_1) \mid x_0 + x_1 = 1\}$

Then we have a simplicial structure $\Delta^0 \rightrightarrows \Delta^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \Delta^2 \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots$ pt

This induces
$$\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \underset{\substack{\| \\ \{\text{codim } p \text{ cycles which intersect each facet properly}\}}}{C^p(X\times\Delta^2)'} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} C^p(X\times\Delta^1)' \rightrightarrows C^p(X\times \overset{\substack{pt\\ \|}}{\Delta^0})$$

$CH^p(X;q) := q^{th}$ homology of this complex.

* $CH^p(X;0) = CH^p(X)$

* <u>Fact</u>: Can replace "simplicial Δ's" by "cubical" ones $\square^0 \rightrightarrows \square^1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \square^2 \cdots$

* $$CH^p(X;1) = \frac{\{\text{finite sums } \sum (Z_i; f_i),\ \text{with codim } Z_i = p-1,\ f_i \in \mathrm{Mero}(Z_i) \text{ s.t. } \sum \mathrm{div}(f_i) = 0\}}{\left\{\begin{array}{l} W \subseteq X \text{ codim } p-2,\ f,g \in \mathrm{Mero}(W), \\ \rightsquigarrow \displaystyle\sum_{\substack{Z\subseteq W\\ \text{codim} 1}} \begin{cases} \text{if } \mathrm{ord}_Z(f)>0, \mathrm{ord}_Z(g)=0 \rightsquigarrow \mathrm{ord}_Z(f)\cdot (Z; -g|_Z) \\ \text{if } \mathrm{ord}_Z(g)>0, \mathrm{ord}_Z(f)=0 \rightsquigarrow -\mathrm{ord}_Z(g)\cdot (Z; -f|_Z) \\ \text{if } \mathrm{ord}_Z(f), \mathrm{ord}_Z(g)>0 \rightarrow \left(Z; (-1)^{\mathrm{ord}_Z(f)\mathrm{ord}_Z(g)} \cdot \frac{f^{\mathrm{ord}_Z(g)}}{g^{\mathrm{ord}_Z(f)}}\Big|_Z\right) \end{cases} \end{array}\right.}$$

"tame symbol"

<u>Definition 1</u>: $CH^p(X;j) =: H_{\mathcal{M}}^{2p-j}(X, \mathbb{Q}(p))$

<u>Alternative definitions</u>:

<u>Note</u>: For X smooth quasi-projective, we have an isomorphism

$$\bigoplus_p CH^p(X)\otimes\mathbb{Q} \xrightarrow{\sim} K_0(X)\otimes\mathbb{Q} \quad \text{K_0-group of } \mathrm{Coh}(X)$$

$$Z \longmapsto [\mathcal{O}_Z]$$

$$\text{Chern character of } E \longleftarrow E$$

In general, we have an isomorphism $\bigoplus_p CH^p(X;j) \xrightarrow{\sim} K_j(X)\otimes\mathbb{Q}$

Beilinson used Quillen's K-theory to define regulator map.

<u>Rmk</u>: For Beilinson's conjecture, one needs to use $\mathrm{Im}\left(K_j^{(p)}(\mathcal{X}) \to K_j^{(p)}(X)\right)$ as suggested by the case of units.

Numerology for Beilinson conjecture:

In practice, classes in $CH^p(X;j)$ are constructed by taking subvarieties $Y=\bigcup Y_j$ of codim d.

$$Y^\circ = Y - \bigcap_{j\neq j'}(Y_j\cap Y_{j'})$$

$$\begin{tikzcd}
\underbrace{CH^1(Y^\circ;1)\otimes\cdots\otimes CH^1(Y^\circ;1)}_{r \text{ copies}} \arrow[rr, "\cup"] \arrow[d, "\text{ét}"'] & & CH^r(Y^\circ;r) \arrow[dl] \\
& & CH^r(Y;r) \arrow[u] \arrow[d] \arrow[dl] \\
H^1_{et}(Y^\circ,\mathbb{Q}_\ell(1))\otimes\cdots\otimes H^1_{et}(Y^\circ,\mathbb{Q}_\ell(1)) \arrow[r, "\cup"] & H^r_{et}(Y^\circ,\mathbb{Q}_\ell(r)) & CH^{r+d}(Y;r) \\
& H^r_{et}(Y,\mathbb{Q}_\ell(r)) \arrow[u] \arrow[d] & \\
& H^{r+2d}_{et}(Y,\mathbb{Q}_\ell(r+d)) &
\end{tikzcd}$$

$CH^1(Y^\circ;1) = \mathcal{O}(Y^\circ)^\times$

Take $\sum_i f_i^{(1)}\cup\cdots\cup f_i^{(r)}$ so that this extends to a class from $CH^r(Y;r)$

$$\text{weight} = (r+2d-1)-2(r+d) = -r-1.$$

$$\text{Center for F.E.} = \frac{w+1}{2} = -\frac{r}{2}$$

i.e. the point for Beilinson conjecture is $r\cdot\frac{1}{2}$ right of the center of F.E.

Generalization of Birch and Swinnerton-Dyer conjecture

When $w=i-2n=-1$, F.E. has center at $s=0$.

$$H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n)) = CH^n(X)\otimes\mathbb{Q}.$$

Put $CH^n(X)_0 := \mathrm{Ker}\left(CH^n(X)\otimes\mathbb{Q} \xrightarrow{\text{class}_{dR}} H^{2n}_{dR}(X/\mathbb{Q})\right)$

Conjecture: $\mathrm{ord}_{s=0} L(H^{2n-1}(X,\mathbb{Q}(n)),s) = \dim CH^n(X)_0\otimes\mathbb{Q}$

Moreover, there is a height pairing $CH^n(X)_0\times CH^n(X)_0 \xrightarrow{ht} \mathbb{R}$

$$L^*(H^{2n-1}(X,\mathbb{Q}(n))) \in \det(\alpha_M)\cdot\det(ht)\cdot\mathbb{Q}^\times.$$