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Periods of modular forms

§1 Eichler-Shimura isomorphism

Let $X = X_1(N)$ be a projective smooth modular curve $/\mathbb{Q}$

$$\begin{tikzcd} \mathcal{E} \arrow[d] & \mathcal{E}^{sm} \arrow[r, hook] \arrow[d] & \bar{\mathcal{E}} \arrow[dl, "\pi"] & D \arrow[l, hook'] \arrow[d] \\ Y \arrow[r, hook] & X \arrow[u, bend left, "\iota"] & & C \arrow[ll] \end{tikzcd}
\qquad
\begin{tikzcd} 0 \arrow[r] & \omega \arrow[r] \arrow[d, phantom, "\|"] & \mathcal{H}^1_{dR,\log} \arrow[r] \arrow[d, phantom, "\|"] & \omega^{-1} \arrow[r] & 0 \\ & \iota^*\Omega^1_{\mathcal{E}^{sm}/X} & R^1\pi_*(\mathcal{O}_{\bar{\mathcal{E}}} \to \Omega^1_{(\bar{\mathcal{E}},D)/(X,C)}) & & \end{tikzcd}$$

$\rightsquigarrow$ gives rise to a filtration on $\mathcal{H}^1_{dR,\log}$:

- $\mathrm{Fil}^{>1}\mathcal{H}^1_{dR,\log} = 0$
- $\mathrm{Fil}^{1}\mathcal{H}^1_{dR,\log} = \omega$ Hodge type $(0,1)$ $(1,0)$
- $\mathrm{Fil}^{\leq 0}\mathcal{H}^1_{dR,\log} = \mathcal{H}^1_{dR,\log}$

$\rightsquigarrow \nabla_{GM} : \mathcal{H}^1_{dR,\log} \longrightarrow \mathcal{H}^1_{dR,\log} \otimes \Omega^1_X(\log C)$ Gauss-Manin connection

induces the Kodaira-Spencer isomorphism $\omega^{\otimes 2} \xrightarrow{\sim} \Omega^1_X(\log C)$

For $k \geq 2$, ∇_{GM} induces a connection

$$\nabla_{GM} : \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log} \longrightarrow \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log} \otimes \Omega^1_X(\log C)$$

$$\begin{tikzcd} \omega^{k-2} \arrow[d, dash] \arrow[dr, "\nabla_{k-2}"] & \omega^{k-2} \otimes \Omega^1_X(\log C) \arrow[d, dash] \\ \omega^{k-4} \arrow[d, dash] \arrow[dr, "\nabla_{k-4}"] & \omega^{k-4} \otimes \Omega^1_X(\log C) \arrow[d, dash] \\ \vdots \arrow[d, dash] \arrow[dr, "\nabla_{2-k}"] & \vdots \arrow[d, dash] \\ \omega^{2-k} & \omega^{2-k} \otimes \Omega^1_X(\log C) \end{tikzcd}$$

Fil^{k-2}, Fil^{k-3}, $\cdots$, Fil^0

∇_{GM} satisfies Griffith transversality:

$$\nabla_{GM}(\mathrm{Fil}^i \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log}) \subseteq \mathrm{Fil}^{i-1}\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log} \otimes \Omega^1_X(\log C)$$

(Griffith transversality is true in all geometric setup

Maybe better to think of $\Omega^1_X(\log C)$ in degree 1 $\rightsquigarrow \nabla_{GM}$ preserves filtrations)

• Fact: $\nabla_{k-2}, \nabla_{k-4}, \cdots, \nabla_{2-k}$ are all isomorphisms.

Thus, given a local section x of ω^{2-k}, lift it uniquely to a section $\tilde{x}$ of $\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log}$

s.t. (inductively) $\nabla_{GM}(\tilde{x})$ has trivial image in $\omega^{2-k}\otimes\Omega^1_X(\log C)$

has trivial image in $\omega^{4-k}\otimes\Omega^1_X(\log C)$

....

$\rightsquigarrow \exists$ a unique lift $\tilde{x}$ s.t. $\nabla_{GM}(\tilde{x}) \in \omega^{k-2}\otimes\Omega^1_X(\log C)$

Denote $\theta: \omega^{2-k} \to \omega^{k-2}\otimes\Omega^1_X(\log C)$ given by $x \mapsto \nabla_{GM}(\tilde{x})$

Conclusion:
$$\begin{array}{ccc} [\omega^{2-k} & \xrightarrow{\theta} & \omega^{k-2}\otimes\Omega^1_X(\log C)] \\ \big\downarrow{\scriptstyle x\mapsto\tilde{x}} & & \big\downarrow \\ DR^\bullet_k := [\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log} & \longrightarrow & \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR,\log}\otimes\Omega^1_X(\log C)] \end{array}$$
is a quasi-isomorphism

Then $H^1(X, DR^\bullet_k) \cong H^1(X, \omega^{2-k} \xrightarrow{\theta} \omega^{k-2}\otimes\Omega^1_X(\log C))$

$$E_1^{p,q} = \begin{array}{|lcl} H^1(X,\omega^{2-k}) & \xrightarrow{\theta} & H^1(X,\omega^{k-2}\otimes\Omega^1_X(\log C)) \\ H^0(X,\omega^{2-k}) & \xrightarrow{\theta} & H^0(X,\omega^{k-2}\otimes\Omega^1_X(\log C)) \\ \hline \end{array} \Rightarrow H^*(X, DR^\bullet_k)$$

Fact: The spectral sequence degenerates at E_1.

$$0 \to H^0(X,\omega^{k-2}\otimes\Omega^1_X(\log C)) \to H^1(X, DR^\bullet_k) \to \underbrace{H^1(X,\omega^{2-k})}_{} \to 0$$

$$H^0(X,\omega^{k-2}\otimes\Omega^1_X(\log C)) = H^0(X,\omega^k) = M_k(\Gamma_1(N))$$

$$H^1(X, DR^\bullet_k) \cong H^1(X(\mathbb{C})^{an}, j_*\mathrm{Sym}^{k-2}\mathcal{H}^1_B(\mathcal{E}/Y))$$

$$H^1(X,\omega^{2-k}) = H^0(X,\omega^{k-2}\otimes\Omega^1_X)^\vee = S_k(\Gamma_1(N))^\vee$$

Hodge decomposition: $H^1(X, DR^\bullet_k) = H^0(X,\omega^k) \oplus \overline{H^0(X,\omega^k(-D))}$

This is called the Eichler-Shimura isomorphism

Consider the Hecke action $\Rightarrow$ if f is an eigen new cuspform of level $\Gamma_1(N)$,

$$0 \to \underbrace{\mathbb{C}\cdot f}_{\mathrm{Fil}^{k-2}M(f)_{dR}} \to \underbrace{H^1(X, DR^\bullet_k)_{\pi_f}}_{M(f)_{dR}} \to \underbrace{\mathbb{C}\cdot\bar{f}}_{\mathrm{Fil}^0 M(f)_{dR}.} \to 0$$

§2 Motive attached to modular forms

Consider Kuga-Sato variety

$$\begin{tikzcd} KS^{sm}_{k-2} = \mathcal{E}^{sm} \times_X \mathcal{E}^{sm} \times \cdots \times_X \mathcal{E}^{sm} \arrow[r, hook] \arrow[d, "\pi_{k-2}"'] & \overline{KS}_{k-2} \arrow[dl] \\ X & \end{tikzcd}$$

(ignore cusp, more complicated)

Step 1: For each $\mathcal{E} \longrightarrow X$, cut out relative $H^1(\mathcal{E}/X)$

Inside

$$\begin{tikzcd} \mathcal{E} \times \mathcal{E} \arrow[d] & \mathcal{E}\times\mathcal{E} \arrow[l, phantom] \arrow[r, hookleftarrow] & \Delta_{\mathcal{E}}, \mathcal{E}\times 0, 0 \times \mathcal{E} \arrow[d] \\ X \times X & & \Delta_X \arrow[ll] \end{tikzcd}$$

Use $pr_1 := [\Delta_{\mathcal{E}}] - [\mathcal{E}\times 0] - [0 \times \mathcal{E}] \in [\mathcal{E}\times\mathcal{E}]$ on $H^*(\mathcal{E})$ to pick out $H^1(\mathcal{E}/X)$

Step 2: Consider $\mathfrak{S}_{k-2} \circlearrowright KS_{k-2} \to X$ (but note H^1 is in degree 1 there's a sign when commuting H^1's.)

Let $sgn: \mathfrak{S}_{k-2} \longrightarrow \{\pm 1\}$ be the sign character.

Then put $H^1(X, \mathrm{Sym}^{k-2} H^1(\mathcal{E}/X)) := sgn_* \circ [pr_1]^{\otimes k-2} (H^{k-1}(KS_{k-2}))$

Step 3: Apply Hecke operators to cut out the motive associated to an eigen new cuspform f.

E.g. $M_k(\Gamma_1(N)) = \mathbb{C}\cdot \overset{=f}{f_0} \oplus \mathbb{C}\cdot f_1 \oplus \mathbb{C} f_2 \oplus \mathbb{C} f_3 \oplus \mathbb{C} f_4$ ← including Eisenstein series.

	f_0	f_1	f_2	f_3	f_4
T_{ℓ_1}-eigenval	α	α	β	β	γ
T_{ℓ_2}-eigenval	α'	β'	γ'	γ'	δ'

↑ new form has mult 1. $\quad$ $\underbrace{\gamma' \quad \gamma'}_{\text{not new form}}$

$$\text{Define } M(f) := \frac{(T_{\ell_1} - \beta)(T_{\ell_1} - \gamma)(T_{\ell_2} - \beta')}{(\alpha - \beta)(\alpha - \gamma)(\alpha' - \beta')} H^1(X, \mathrm{Sym}^{k-2} H^1(\mathcal{E}/X))$$

Fact: $M(f)_B \cong H^1(X, j_* \mathrm{Sym}^{k-2} R^1\pi_{1*} \mathbb{Q}_{\mathcal{E}/Y})$

$M(f)_{dR} \cong H^1(X, \mathrm{Sym}^{k-2} \mathcal{H}^1_{dR,\log} \longrightarrow \mathrm{Sym}^{k-2} \mathcal{H}^1_{dR,\log} \otimes \Omega^1_X(\log C))$

§3. L-functions associated to modular forms.

Let $f(\tau)=\sum_{n\geq 0} a_n q^n$ be a normalized eigen newform of level $\Gamma_1(N)$ and weight k.

nebentypus character $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

$\Rightarrow \forall \ell \nmid N$, T_ℓ-eigenvalue $= a_\ell$, $\quad T_\ell(f)=\left(\sum_{n\geq 0} a_{\ell n} q^\ell\right)+\ell^{k-1}\chi(\ell)\sum_{n\geq 0} a_n q^{n\ell}$

$\forall \ell \mid N$, U_ℓ-eigenvalue $= a_\ell$ $\quad U_\ell(f)=\sum_{n\geq 0} a_{\ell n} q^\ell$.

$M(f)_{et,p}$ = rank 2 representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ weight $= 1-k$. $p\nmid N$

when $\ell \nmid Np$, char poly of geometric Frobenius ϕ_ℓ is $x^2 - a_\ell x + \ell^{k-1}\chi(\ell)$

$\ell = p$, the char poly of crystalline Frobenius ϕ_p on $D_{cris}(M_{et,p})$ is $x^2 - a_p x + p^{k-1}\chi(p)$

$$L(f,s)=L(M(f)_{et,p},s)=\prod_{\ell\nmid N}\frac{1}{\det\left(id-\underline{\rho_p(\phi_\ell)}\,\ell^{-s}\right)}\cdot\prod_{\ell\mid N}(\cdots)$$

when $\ell=p$, use $D_{cris}(M_{et,p})$ instead

$$=\prod_{\ell\nmid N}\frac{1}{1-a_\ell \ell^{-s}+\chi(\ell)\ell^{k-1-s}}\prod_{\ell\mid N}\frac{1}{1-a_\ell \ell^{-s}}$$

$$=\sum_{n\geq 1}\frac{a_n}{n^s}$$

- $\Lambda(f,s)=(2\pi)^{-s}\Gamma(s)\cdot L(f,s)$.

Interpretation of $\Lambda(f,s)$ via integration (when f is cuspidal) : $q=e^{2\pi i\tau}$, $\tau=u+iv$

$$\int_0^\infty f(iv)v^s\frac{dv}{v}=\int_0^{+\infty}\sum_{n\geq 1}a_n e^{-2n\pi v}\cdot v^s\frac{dv}{v}=\sum_{n\geq 1}a_n\cdot\int_0^{+\infty}e^{-2n\pi v}\cdot v^s\cdot\frac{dv}{v}$$

$$\overset{w=2n\pi v}{=\!=}\sum_{n\geq 1}a_n\cdot\frac{1}{(2n\pi)^s}\underbrace{\int_0^{+\infty}e^{-w}w^s\frac{dw}{w}}_{\Gamma(s)}=(2\pi)^{-s}\Gamma(s)\sum_{n\geq 1}\frac{a_n}{n^s}$$

(Assume that f has level 1, we see analytic continuation $\Rightarrow$ k even)

$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\in SL_2(\mathbb{Z}) \rightsquigarrow f\left(-\frac{1}{z}\right)=z^k\cdot f(z) \Rightarrow f\left(\frac{i}{v}\right)=(-1)^{\frac{k}{2}}v^k f(iv)$

Then $\int_0^{+\infty} f(iv)v^s \frac{dv}{v} = \int_1^{+\infty} f(iv)\, v^s \frac{dv}{v} + \underbrace{\int_0^1 (-1)^{\frac{k}{2}} v^{-k} f\left(\frac{i}{v}\right) \cdot v^s \frac{dv}{v}}_{\shortparallel}$

$$(-1)^{\frac{k}{2}} \int_1^{\infty} v^k f(iv) \cdot v^{-s} \frac{dv}{v}$$

functional equation $s \leftrightarrow k-s$

$$\boxed{\Lambda(f,s) = (-1)^{\frac{k}{2}} \Lambda(f,k-s)}$$

§4. Periods of modular forms and critical L-values

$\Lambda^2 M(f) \simeq \mathbb{Q}(1-k)$

- $M = M(f)_B$ rank 2, $M(n)_B^+ = 1$-dim'l. (say, q-expansion of f has coeffs in $\mathbb{Q}$)

$M_{dR} = M(f)_{dR}$ rank 2, Hodge type $(0,k-1), (k-1,0)$

$\alpha_{M(n)}^{\mathbb{R}} : M(n)_B^+ \otimes \mathbb{R} \longrightarrow M_{dR}/F^n M_{dR} \otimes \mathbb{R}$

When $n \in [1,k-1]$, $\dim F^n M_{dR} = 1. \rightsquigarrow \alpha_{M(n)}^{\mathbb{R}}$ is an isomorphism.

$\Rightarrow L(f,n)$ is critical. <u>Expected</u>: $L(f,n) \in \mathbb{Q}^\times \cdot \det(\alpha_{M(n)}^{\mathbb{R}})$

Write $M_{dR} = \mathbb{Q}\omega_f \oplus \mathbb{Q}\eta_f$, ω_f basis of $F^{k-1}M_{dR}$, η_f complementary basis, not canonical

$$M_{dR} \otimes \mathbb{C} \simeq M_B \otimes \mathbb{C} \qquad M_B = \underset{\substack{\circlearrowleft \\ F_\infty = 1}}{\mathbb{Q}e_f^+} \oplus \underset{\substack{\circlearrowleft \\ F_\infty = -1}}{\mathbb{Q}e_f^-}, \quad \omega_f = \Omega_f^+ e_f^+ + \Omega_f^- e_f^-$$

<u>normalization</u>: $\omega \wedge \bar{\eta} = \int \omega \wedge \bar{\eta} \cdot e_B^{top}$ Always use $\Lambda^{top} M_{dR}$ as basis

Poincaré pairing: $M_{dR}/F^n M_{dR} \times \underbrace{F^{k-n} M_{dR}}_{} \longrightarrow \Lambda^2 M_{dR} = \mathbb{Q}$ perfect pairing

$$\eta_f \times \omega_f \longmapsto 1.$$

$$M(n)_B^+ \quad \langle (2\pi i)^n e_f^{(-1)^n}, \underset{\substack{\shortparallel \\ \Omega_f^+ e_f^+ + \Omega_f^- e_f^-}}{\omega_f} \rangle = (2\pi i)^{n-(k-1)} \cdot \Omega_f^{(-1)^{n+1}}$$

$\det \alpha_{M(n)} = (2\pi i)^{n-(k-1)} \cdot \Omega_f^{(-1)^{n+1}}$

On the other hand, write e_f^{+*}, e_f^{-*} for the dual basis of e_f^+, e_f^-

$e_f^{\pm,*}$ can be represented by a class in $H_1(X \text{ rel } C, j_* \mathrm{Sym}^{k-2} H_1^B(\mathcal{E}/Y))_f^{\pm}$

e.g. path $\{0, i\infty\} \otimes (S^{*\otimes(k-2-m)} \otimes T^{*\otimes m})$ sgn is $(-1)^m$

$\circlearrowright$ $F_\infty = 1$ $\circlearrowright$ $F_\infty = (-1)^m$

Then $\langle \omega_f, e_f^{*,\pm} \rangle \sim \Omega_f^{\pm}$ ← sign $= (-1)^m$

$$\| $$

$$\int_0^{i\infty} f(\tau) \otimes \langle \underbrace{(2\pi i\, dz)^{\otimes k-2}, S^{*\otimes(k-2-m)} \otimes T^{*\otimes m}}_{\tau^m} \rangle \cdot 2\pi i\, d\tau$$

$$= (2\pi i)^{k-1} \cdot L(f, m+1) \cdot (2\pi)^{-(m+1)} \Gamma(m+1)$$

$$\Rightarrow \det \alpha_{M(n)} = (2\pi i)^{n-(k-1)} \Omega_f^{(-1)^{n+1}} = (2\pi i)^{n-(k-1)} \cdot (2\pi i)^{k-1} \cdot (2\pi)^{-(m+1)} \Gamma(m+1) L(f, m+1)$$

$$\sim (2\pi)^{n-(m+1)} L(f, m+1)$$

When $n = m+1$, okay, and also $L(f, n) \sim (2\pi i)^{n-(k-1)} \Omega_f^{(-1)^n}$