Special values of L-functions 22

—— Examples of Beilinson's conjecture I

§1 Rankin-Selberg motive and dimension numerology

Recall for f eigenform of weight k, $M(f)$ is pure of weight $k-1$

of Hodge type $(0,k-1)$, $(k-1,0)$

F.E. for f is $\Lambda(f,s) = \varepsilon(f,s)\cdot\Lambda(f^\vee, k-s)$.

$$\alpha_M^{\mathbb{R}} : M(f)(n)_B^+ \otimes \mathbb{R} \longrightarrow M(f)_{dR}/F^n M(f)_{dR}$$

is an isomorphism if $n \in [1, k-1]$

But when $n=k$, $\mathrm{Coker}(\alpha_M^{\mathbb{R}}) = 1$-dim'l.

When $k=2$, $M(f)(2)$ appears in $H^1(X,\mathbb{Q}(2))$ for X modular curve.

Beilinson conjecture expects $\exists$ $H^2_M(X,\mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_\mathcal{D}} H^2(X,\mathbb{R}_\mathcal{D}(2)) = \mathrm{Coker}\,\alpha_M^{\mathbb{R}}$.

Will discuss this later. $\qquad \overset{\cup}{\theta\otimes\theta} \longmapsto$ related to $L(f,2)$.

Consider $f\in S_k(\Gamma_0(N))$ and $g \in S_\ell(\Gamma_0(N))$ new eigenforms of weight $\ell<k$.

We may talk about the motive $M(f)\otimes M(g)$

Hodge type: $\{(0,k-1),(k-1,0)\}\otimes\{(0,\ell-1),(\ell-1,0)\}$ $\quad$ pure of wt $(k-1)+(\ell-1)$

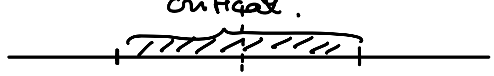
$$= \{(0,k+\ell-2),(\ell-1,k-1),(k-1,\ell-1),(k+\ell-2,0)\}$$

$$\alpha^{\mathbb{R}}_{M(f)\otimes M(g)} : \underbrace{(M(f)\otimes M(g)(n))_B^+}_{2\text{-dim'l}} \otimes\mathbb{R} \longrightarrow \underbrace{(M(f)\otimes M(g))_{dR}/F^n(M(f\times g)_{dR})}_{\text{This is 2-dim'l if } \ell\le n\le k-1} \otimes\mathbb{R}$$

E.g. when $k=\ell$, NO critical values n ! (for Deligne's conjecture)

When $k>\ell$, functional equation for $L(f\times g,s)$ has center at $\frac{k+\ell-1}{2}$

critical.

critical values 

$F_\infty = 1$ $\quad F_\infty=-1$ $\quad \ell \quad \frac{k+\ell-1}{2} \quad k-1$

Write $M(f)_B = \mathbb{Q} e_f^+ \oplus \mathbb{Q} e_f^-$, $M(f)_{dR} = \mathbb{Q}\cdot\omega_f \oplus \mathbb{Q}\eta_f$ (but η_f not canonical).

ω_f in Fil^{k-1}

Similar for $M(g)_B$ and $M(g)_{dR}$.

In the critical case, $\underbrace{M(f\times g)(n)_B^+}\otimes\mathbb{R} \xrightarrow[\simeq]{\alpha_{M(n)}} M(f\times g)_{dR}/F^n$

To compute: basis $(2\pi i)^n e_f^+\otimes e_g^{(-1)^n}, (2\pi i)^n e_f^-\otimes e_g^{-(-1)^n}$ $\qquad$ basis $\eta_f\otimes\omega_g, \eta_f\otimes\eta_g$

Write $(\omega_f, \eta_f) = (e_f^+ \ e_f^-)\underbrace{\begin{pmatrix}\Omega_f^+ & \Theta_f^+\\ \Omega_f^- & \Theta_f^-\end{pmatrix}}_{}$ $\qquad (\omega_g\ \eta_g) = (e_g^+\ e_g^-)\underbrace{\begin{pmatrix}\Omega_g^+ & \Theta_g^+\\ \Omega_g^- & \Theta_g^-\end{pmatrix}}_{}$

$\det = (2\pi i)^{k-1}$ b/c $\wedge^2 M(f) = \mathbb{Q}(1-k)$ $\qquad \det = (2\pi i)^{\ell-1}$

$$\Rightarrow (e_f^+\ e_f^-) = (2\pi i)^{1-k}\cdot(\omega_f\ \eta_f)\begin{pmatrix}\Theta_f^- & -\Theta_f^+\\ -\Omega_f^- & \Omega_f^+\end{pmatrix},\ \text{same for } g$$

So
$$\det\alpha_{M(n)} = (2\pi i)^{2n-2(k-1)-2(\ell-1)}\det\begin{pmatrix}-\Omega_f^-\,\Theta_g^\pm & \Omega_f^+\cdot(-\Theta_g^\mp)\\ -\Omega_f^-\cdot(-\Omega_g^\pm) & \Omega_f^+\cdot\Omega_g^\mp\end{pmatrix}$$
$$\sim (2\pi i)^{2n-2(k-1)-(\ell-1)}\cdot\Omega_f^+\Omega_f^-.$$

The period is independent of the form with smaller weight!!!

Recall $\omega_f = \Omega_f^+ e_f^+ + \Omega_f^- e_f^- \Rightarrow \bar\omega_f = \Omega_f^+ e_f^+ - \Omega_f^- e_f^- \in M(f)_{dR}\otimes\mathbb{C}$

Thus $\omega_f\wedge\bar\omega_f = 2\Omega_f^+\Omega_f^- e_f^+\wedge e_f^- \in \wedge^2 M(f)_{dR}\otimes\mathbb{C} \simeq \wedge^2 M(f)_B\otimes\mathbb{C}$

For modular forms, this means that

$$2\Omega_f^+\Omega_f^- = \int_{Y_0(N)} f(\tau)\otimes(2\pi i dz)^{\otimes k-2}\cdot 2\pi i\, d\tau \cdot \overline{f(\tau)\otimes 2\pi i\, dz^{\otimes k-2}}\cdot 2\pi i\, d\bar\tau$$

$$\sim_{\mathbb{Q}^\times} (2\pi i)^{2(k-1)}\cdot\int_{Y_0(N)} f(\tau)\overline{f(\tau)}\cdot v^{k-2}\, du\wedge dv = (2\pi i)^{2(k-1)}\langle f,f\rangle$$

So $\det\alpha_{M(n)} \sim (2\pi i)^{2n-(\ell-1)}\langle f,f\rangle$

$dz = S+\tau T$, $d\bar z = S+\bar\tau T$

$dz\wedge d\bar z = (\tau-\bar\tau)\underbrace{T\wedge S}_{\text{top Betti class}}$

Beilinson conjecture starts when $n=k$

Case of $l=k=2$, Hodge types are $(0,2)$, $(1,1)\times 2$, $(2,0)$

$$\alpha_{M(2)}: \underbrace{M(f\times g)_B^+}_{2\text{-dim'l}} \longrightarrow \underbrace{M(f\times g)_{dR}/F^2M(f\times g)_{dR}}_{3\text{-dim'l.}} \quad \text{Coker has dim}=1.$$

Secret of the trade: can only expect to construct motivic elements when Coker $=1$.

b/c only rank 1 case, regulator $\leftrightarrow$ L-values.

$M(f\times g)(2)$ appears in $H^2(X\times X, \mathbb{Q}(2))$ for X = modular curve.

So expect the motivic class in $H^3_{\mathcal{M}}(X\times X, \mathbb{Q}(2))$

This comes from diagonal embedding $Y \subseteq X \xrightarrow{\Delta} X\times X$

$$\begin{array}{ccc} H^1_{\mathcal{M}}(Y,\mathbb{Q}(1)) & \xrightarrow{\text{Gysin}} & H^3_{\mathcal{M}}(Y\times Y,\mathbb{Q}(2)) & \\ \parallel & & \uparrow \leftarrow \text{class extends to } X & \\ \mathcal{O}(Y)^\times_{\mathbb{Q}} & & H^3_{\mathcal{M}}(X,\mathbb{Q}(2)) & \xrightarrow{reg_{\mathcal{D}}} H^3(X,\mathbb{R}_{\mathcal{D}}(2)). \end{array}$$

New cases? Unitary Shimura variety $U(1,n-1)$ / E = imaginary quadratic field

Consider a "nice" automorphic rep'n π appearing in $H^{n-1}(Sh_{U(1,n-1)},\mathbb{Q})$

Associated Galois rep'n $\rho_\pi: Gal_E \to GL_n(\overline{\mathbb{Q}}_\ell)$ Hodge-Tate weights $0,1,\dots,n-1$ at each embedding

Consider $M(\pi)$ whose étale realization is $\mathrm{Ind}_{Gal_E}^{Gal_{\mathbb{Q}}}\rho_\pi$, So HT weights $(0,1,\dots,n-1)\times 2$

When n odd, $\underbrace{M(\pi)_B^+}_{n\text{-dim'l}} \longrightarrow \underbrace{M(\pi)_{dR}/F^{\frac{n+1}{2}}M(\pi)_{dR}}_{\text{see } (0,1,\dots\frac{n-1}{2})\times 2 \text{ totally } (n+1)\text{-dim'l}.}$ ✓

§2 Rankin-Selberg integral

Consider $f\in M_k(\Gamma_0(p))$ and $g\in M_\ell(\Gamma_0(p))$ normalized new eigenforms of weight $\ell<k$. Assume at least one of f & g is a cusp form

(For simplicity, we assume that the level p is a prime.)

We give an integral formula for $L(f\times g,s)$ (This works for $GL(m)\times GL(n)$.)

Consider the non-holomorphic Eisenstein series

$$E_{k+2s,-s}(\tau,0)=\pi^{-(k+s)}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{(\operatorname{Im}\tau)^s}{|m\tau+n|^{2s}(m\tau+n)^k}$$

later will take $k=wt(f)-wt(g)$

weight $=k$

(This differs from earlier def'n by $(2i)^k$)

Think of $\{(c,d)\text{ coprime}\}$ as $\overset{\Gamma_\infty}{\overbrace{\begin{pmatrix}1&\mathbb{Z}\\&1\end{pmatrix}}}\backslash SL_2(\mathbb{Z})$, but we need $\begin{pmatrix}1&\mathbb{Z}\\&1\end{pmatrix}\backslash\Gamma_0(p)$

$$(c,d)\longleftarrow\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

Define $E^{(p)}_{k+2s,-s}(\tau,0)=p^{-s}E_{k+2s,-s}(p\tau,0)-p^{-2s+k}\cdot E_{k+2s,-s}(\tau,0)$

$$=\pi^{-(k+s)}\cdot\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{p^{-s}\cdot(\operatorname{Im}(p\tau))^s}{|mp\tau+n|^{2s}(mp\tau+n)^k}-\frac{(\operatorname{Im}\tau)^s}{|mp\tau+pn|^{2s}(mp\tau+pn)^k}\right)$$

$$=\pi^{-(k+s)}\cdot\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ p\nmid n}}\frac{(\operatorname{Im}\tau)^s}{|mp\tau+n|^{2s}(mp\tau+n)^k}$$

$$=\pi^{-(k+s)}\cdot\left(\sum_{p\nmid d}\frac{1}{d^{k+2s}}\right)\cdot\sum_{\substack{(m,n)=1\\ p\mid m,\,p\nmid n}}\frac{(\operatorname{Im}\tau)^s}{|m\tau+n|^{2s}(mp\tau+n)^k}$$

$$=\pi^{-(k+s)}\cdot\zeta^{(p)}(k+2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)}\frac{(\operatorname{Im}\gamma\tau)^s}{j(\gamma,\tau)^k}$$

where for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $j(\gamma,\tau)=c\tau+d$.

Consider the integral $I(f,g,s)=\displaystyle\int_{\Gamma_0(p)\backslash\mathfrak{h}}\overline{f}\cdot g\,E_{k-l+2s,-s}(\tau,0)(\operatorname{Im}\tau)^k\cdot\frac{du\,dv}{v^2}$ $\qquad\tau=u+iv$.

$$=\pi^{l-k-s}\cdot\zeta^{(p)}(k-l+2s)\int_{\Gamma_0(p)\backslash\mathfrak{h}}\overline{f}(\tau)\cdot g(\tau)\cdot\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)}\frac{(\operatorname{Im}\gamma\tau)^s}{j(\gamma,\tau)^{k-l}}\cdot(\operatorname{Im}\tau)^k\cdot\frac{du\,dv}{v^2}$$

$$=\pi^{l-k-s}\cdot\zeta^{(p)}(k-l+2s)\int_{\Gamma_0(p)\backslash\mathfrak{h}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)}\frac{\overline{f}(\gamma\tau)}{\cancel{\overline{j(\gamma,\tau)}^k}}\cdot\frac{g(\gamma\tau)}{\cancel{j(\gamma,\tau)^l}}\cdot\frac{(\operatorname{Im}\gamma\tau)^s}{\cancel{j(\gamma,\tau)^{k-l}}}\cdot(\operatorname{Im}\gamma\tau)^k\cdot\cancel{|j(\gamma,\tau)|^{2k}}\frac{du\,dv}{v^2}$$

unfolding
$$= \pi^{\ell-k-s} \cdot \zeta^{(p)}(k-\ell+2s) \int_{\Gamma_\infty \backslash \mathfrak{h}} \bar{f}(\tau) \cdot g(\tau) \cdot (\operatorname{Im}\tau)^{k+s} \frac{du\,dv}{v^2}$$

$f = \sum a_m q^m$, $g = \sum_n b_n q^n$

$$= \pi^{\ell-k-s} \cdot \zeta^{(p)}(k-\ell+2s) \int_{v=0}^{+\infty} \sum_{m,n} a_m b_n \underbrace{\int_{u=0}^{1} e^{-2\pi i m(u-iv)} \cdot e^{2\pi i n(u+iv)}}_{\text{require } m=n} \cdot v^{k+s} \frac{du\,dv}{v^2}$$

$$= \pi^{\ell-k-s} \cdot \zeta^{(p)}(k-\ell+2s) \sum_{n\geq 1} \int_{v=0}^{+\infty} a_n b_n e^{-4\pi n v} \cdot v^{k+s} \frac{dv}{v^2}$$

$$= \pi^{\ell-k-s} \cdot \zeta^{(p)}(k-\ell+2s)\, \Gamma(k-1+s) \cdot \sum_{n\geq 1} \frac{a_n b_n}{(4\pi n)^{k-1+s}}$$

Local computation of $\tilde{L}(f\times g, s) := \sum_{n\geq 1} \frac{a_n b_n}{n^s} = \prod_{q \text{ prime}} \left(\sum_{m\geq 0} \frac{a_{q^m} b_{q^m}}{q^{ms}} \right)$

When $q \neq p$, $X^2 - a_q X + q^{k-1} = 0$ has roots $\alpha_1(q)$ and $\alpha_2(q)$

$X^2 - b_q X + q^{k-1} = 0$ has roots $\beta_1(q)$ and $\beta_2(q)$ (omit q from the notation)

Then $\frac{1}{(1-\alpha_1 X)(1-\alpha_2 X)} = 1 + a_q X + a_{q^2} X^2 + \cdots$ $\quad X = q^{-s}$

$\frac{1}{(1-\beta_1 X)(1-\beta_2 X)} = 1 + b_q X + b_{q^2} X^2 + \cdots$

Claim:
$$\frac{1 - \alpha_1\alpha_2\beta_1\beta_2 X^2}{(1-\alpha_1\beta_1 X)(1-\alpha_1\beta_2 X)(1-\alpha_2\beta_1 X)(1-\alpha_2\beta_2 X)} = 1 + a_q b_q X + a_{q^2} b_{q^2} X^2 + \cdots$$

Proof: Write $A_r = a_{q^r}$ and $B_r = b_{q^r}$.

Then $A_0 = 1$, $A_1 = \alpha_1 + \alpha_2$, $A_2 = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2$, $A_3 = \alpha_1^3 + \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 + \alpha_2^3$, $\cdots$

Consider $T_\alpha := \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \circlearrowright V = \mathrm{std}_2$, then $A_n = \operatorname{Tr}(T_\alpha ; \operatorname{Sym}^n V)$

$T_\beta := \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \circlearrowright W = \mathrm{std}_2$, then $B_n = \operatorname{Tr}(T_\beta ; \operatorname{Sym}^n W)$

So $A_n B_n = \operatorname{Tr}(T_\alpha \otimes T_\beta, \operatorname{Sym}^n V \otimes \operatorname{Sym}^n W)$

For similar reason, we have the following expression of generating series

$$\frac{1}{(1-\alpha_1\beta_1X)(1-\alpha_1\beta_2X)(1-\alpha_2\beta_1X)(1-\alpha_2\beta_2X)} = \sum_{n\geq 0} \mathrm{Tr}\left(T_\alpha \otimes T_\beta \,;\, \mathrm{Sym}^n(V\otimes W)\right) X^n$$

This equality follows from the algebraic equality

$$\mathrm{Sym}^n(V\otimes W) = \bigoplus_{m\equiv n(2)} \left(\mathrm{Sym}^m V \otimes \mathrm{Sym}^m W\right) \otimes \left(\wedge^2 V \otimes \wedge^2 W\right)^{\otimes \frac{n-m}{2}}$$

(<u>note</u>: $\wedge^2 V \otimes \wedge^2 W \hookrightarrow \mathrm{Sym}^2(V\otimes W)$ then check dimension.) $\square$

In application $\alpha_1\alpha_2\beta_1\beta_2 = p^{k+l-2}$.

So $$\tilde{L}(f\times g, s) = \frac{L(f\times g, s)}{\zeta^{(p)}(2s-(k+l-2))}$$

So we have $$I(f,g,s) = \pi^{l-k-s} \cdot \cancel{\zeta^{(p)}(k-l+2s)}\,\Gamma(k-1+s)\cdot(4\pi)^{-(k-1+s)} \cdot \frac{L(f\times g, k-1+s)}{\cancel{\zeta^{(p)}(2s+k-l)}}$$

$$= 4^{1-k-s} \cdot \pi^{l-2k+1-2s} \cdot \Gamma(k-1+s) \cdot L(f\times g, k-1+s)$$

<u>Theorem</u> (Shimura) When $m \in [0, k-l-1] \cap \mathbb{Z}$

$$L(f\times g, l+m) \in \pi^{l+2m+1} \cdot \langle f, f\rangle \cdot \mathbb{Q}^\times \sim \det\left(\alpha_{M(f\times g)(l+m)}\right)$$

<u>Sketch</u>: Consider $I(f\times g, l+m) \sim \pi^{l-1-2(l+m)} \cdot L(f\times g, l+m)$

$\wr \qquad\qquad\qquad\qquad\qquad \wr\ \pi^{l+2m+1}\langle f, f\rangle$

$\langle f, g E^{(p)}_{k-l+2s,-s}\big|_{s=l+m+1-k}\rangle$

Suffices to show $\langle f, g E^{(p)}_{k-l+2s,-s}\rangle \sim \langle f, f\rangle$

<u>Upshot</u>: Shimura's differential operator $\delta_k := \frac{1}{2\pi i}\left(\frac{\partial}{\partial \tau} + \frac{k}{\tau-\bar{\tau}}\right) : C^\infty(\omega^k) \to C^\infty(\omega^{k+2})$

<u>Fact</u>: $g E^{(p)}_{k-l+2s,-s} = h_k + \delta_{k-2}(h_{k-2}) + \delta_{k-2}\circ\delta_{k-4}(h_{k-4}) + \cdots + \delta_{k-2}\circ\cdots\circ\delta_{k-2r}(h_{k-2r})$

with each $h_{k-2i} \in S_{k-2i}(\mathbb{Q})$

$\mathbb{Q}$
$\cup$

$\langle f, g E^{(p)}_{k-l+2s,-s}\rangle = \langle f, h_k\rangle = \langle f, f\rangle \cdot$ coeff on f when writing h_k as sum of eigenforms