Special values of L-functions 23

—— Example of Beilinson's conjecture

Plan of this lecture:

(1) Define a function $\theta \in \mathcal{O}(Y)^\times$ for $Y =$ open modular curve

(2) Compute $H^1_{\mathcal{M}}(Y, \mathbb{Q}(1)) \xrightarrow[\Delta]{\text{Gysin}} H^3_{\mathcal{M}}(Y \times Y, \mathbb{Q}(2)) \supseteq H^3_{\mathcal{M}}(X \times X, \mathbb{Q}(2)) \xrightarrow{\text{reg}_{\mathcal{D}}} H^3(X \times X, \mathbb{R}_{\mathcal{D}}(2))$

$$\theta \longmapsto \text{reg}_{\mathcal{D}}(\theta)$$

(3) Compute $H^1_{\mathcal{M}}(Y, \mathbb{Q}(1))^{\otimes 2} \xrightarrow{\text{cup}} H^2_{\mathcal{M}}(Y, \mathbb{Q}(2)) \supseteq H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \xrightarrow{\text{reg}_{\mathcal{D}}} H^2(X, \mathbb{R}_{\mathcal{D}}(2))$

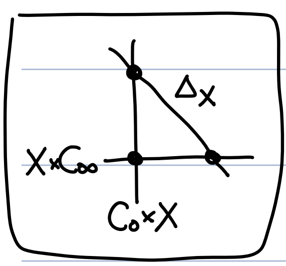
§1 Siegel units:

Proposition (Manin-Mumford) Suppose that X is a modular curve, and D is a degree 0 divisor supported on cusps. Then D is torsion in $\mathrm{Jac}(X)$

We will consider an easy version of this for $X = X_0(p)$

Just take $\theta(\tau) := \Delta(p\tau)/\Delta(\tau)$ $\qquad \Delta =$ Ramanujan function

Then zeros and poles of θ are at the cusps C_0 and C_∞, $\mathrm{ord}_{C_\infty}\theta(\tau) = p-1 \Rightarrow \mathrm{ord}_{C_0}\theta(\tau)$



Consider $\tilde{\Delta}_X := \Delta_X \cup (X \times C_\infty) \cup (C_0 \times X) \subseteq X \times X$

	Δ_X	$X \times C_\infty$	$C_0 \times X$
	θ	θ^{-1}	θ^{-1}
order at $C_0 \times C_0$	$(p-1)$		$-(p-1)$
$C_\infty \times C_\infty$	$-(p-1)$	$(p-1)$	
$C_0 \times C_\infty$		$-(p-1)$	$(p-1)$

So $\theta_{\text{prod}} := (\Delta_X, \theta) + (X \times C_\infty, \theta^{-1}) + (C_0 \times X, \theta^{-1})$ defines an element of $H^3_{\mathcal{M}}(X \times X, \mathbb{Q}(2))$

§2 Regulators

① For $f \in \mathcal{O}(X)^\times = H^1_{\mathcal{M}}(X, \mathbb{Z}(1))$, understand $r_{\mathcal{D}}(f) \in H^1(X, \mathbb{R}_{\mathcal{D}}(1))$

- $\mathbb{R}_{\mathcal{D}}(1) \cong [\mathbb{R}(1) \oplus \Omega_X^{\geq 1} \longrightarrow \Omega_X^\bullet] \cong \mathrm{Cone}(\Omega_X^{\geq 1} \xrightarrow{pr_{\mathbb{R}}} \underline{\mathbb{R}})[-1]$

$$\Omega_X^\bullet \underset{\|\wr}{} \underline{\mathbb{C}} \qquad \cong \mathrm{Cone}(\Omega_X^{\geq 1} \xrightarrow{pr_{\mathbb{R}}} A^\bullet_{X,\mathbb{R}})[-1]$$

$\llcorner$ $\mathbb{R}$-valued C^∞-functions

namely,
$$\begin{tikzcd}
\text{deg } 1 & \text{deg } 2 & \text{deg } 3 \\
\Omega_X^1 \arrow[r, "\partial"] \arrow[dr, "pr_{\mathbb{R}}"] & \Omega_X^2 \arrow[r, "\partial"] \arrow[dr, "pr_{\mathbb{R}}"] & \Omega_X^3 \arrow[r] & \cdots \\
\oplus & \oplus & & \\
A^0_{X,\mathbb{R}} \arrow[r, "d"] & A^1_{X,\mathbb{R}} \arrow[r] & A^2_{X,\mathbb{R}}
\end{tikzcd}$$

So $H^1(X, \mathbb{R}_{\mathcal{D}}(1)) = \{(\eta, \omega) \in C^\infty(X, \mathbb{R}) \oplus \Omega_X^1(X) \mid d\eta = \frac{1}{2}(\omega + \bar{\omega})\}$

Recall $\mathbb{Z}_{\mathcal{D}}(1) = [\mathbb{Z}(1) \longrightarrow \mathcal{O}_X] = \mathcal{O}_X^\times[-1]$

So $\mathcal{O}(X)^\times \xrightarrow{\sim} H^0(X, \mathcal{O}_X^\times) \cong H^1(X, \mathbb{Z}_{\mathcal{D}}(1)) \longrightarrow H^1(X, \mathbb{R}_{\mathcal{D}}(1))$

$f \longmapsto \eta_f = \log|f|, \ \omega_f = \frac{df}{f}$

$$\begin{tikzcd}
 & \mathbb{Z}(1) \arrow[r] \arrow[d] & \underline{\mathbb{R}}(1) \arrow[r] \arrow[d] & \underline{\mathbb{R}}(1) \arrow[d] & & \\
\mathcal{O}_X^\times & \mathcal{O}_X \arrow[l, "\exp"'] \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X \oplus \Omega_X^1 \arrow[r, "pr_{\mathbb{R}} \oplus id"] \arrow[d, "\partial"] \arrow[dl, phantom] & A_X^0 \oplus \Omega_X^1 \arrow[d, "d"] & \\
 & & & \Omega_X^1 \oplus \Omega_X^2 \arrow[r] \arrow[d, "\partial"] & A_X^1 \oplus \Omega_X^2 \arrow[d, "d"] & \\
 & & & \Omega_X^2 \arrow[d] \quad \vdots & A_X^2 \quad \vdots & \\
 & & & \vdots & \vdots &
\end{tikzcd}$$

$f \in \mathcal{O}_X^\times$, "$\log f$" $\in \mathcal{O}_X$, "$\log f$" $\in \mathcal{O}_X$, $\left(\text{"}\log f\text{"}, \frac{df}{f}\right) \in \mathcal{O}_X \oplus \Omega_X^1$, $\left(\log|f|, \frac{df}{f}\right) \in A_X^0 \oplus \Omega_X^1$

② Let S be a projective smooth surface$/\mathbb{Q}$ (e.g. $S = X \times X$)

Regulator map: $reg_{\mathcal{D}} : H^3_{\mathcal{M}}(S, \mathbb{Q}(2)) \longrightarrow H^3(S, \mathbb{R}_{\mathcal{D}}(2))$

where $\mathbb{R}_{\mathcal{D}}(2) = [\underline{\mathbb{R}}(2) \oplus \Omega_S^2[-2] \longrightarrow \Omega_S^\bullet] \cong [\Omega_S^2[-2] \xrightarrow{pr_{\mathbb{R}(1)}} \underline{\mathbb{R}}(1)]$

$$\Omega_S^\bullet \underset{\|\wr}{} \underline{\mathbb{C}}$$

So $H^3(S, \mathbb{R}_{\mathcal{D}}(2))$ sits in the exact sequence

$$\cdots \longrightarrow H^0(S, \Omega_S^2) \xrightarrow{pr_{\mathbb{R}(1)}} H^2(S, \mathbb{R}(1)) \longrightarrow H^3(S, \mathbb{R}_{\mathcal{D}}(2))$$
$$\longrightarrow H^1(S, \Omega_S^2) \hookrightarrow H^3(S, \mathbb{R}(1))$$

So $H^3(S, \mathbb{R}_{\mathcal{D}}(2)) \simeq \dfrac{H^2(S,\mathbb{R}(1))}{pr_{\mathbb{R}(1)} F^2 H^2_{dR}(S)} \cong H^{1,1} \cap H^2(S,\mathbb{R}(1))$

Then there is a natural pairing $H^3(S,\mathbb{R}_{\mathcal{D}}(2)) \times H^{1,1}(X\times X,\mathbb{C}) \longrightarrow \mathbb{C}$

Fact: Suppose that $\sum_j (Y_j, f_j)$ represents a class c in $H^3_{\mathcal{M}}(S,\mathbb{Q}(2))$, and $\omega \in H^{1,1}$

Then $\langle reg_{\mathcal{D}}(c), \omega \rangle = \boxed{\dfrac{1}{2\pi i}} \sum_j \int_{Y_j(\mathbb{C})} \log|f_j| \cdot \omega$

mult by $2\pi i$?

Corollary Let f, g be two normalized cuspidal eigen newform of level $\Gamma_0(p)$ and wt 2

$\rightsquigarrow \omega_f = f(q)\dfrac{dq}{q} = 2\pi i f(\tau)d\tau$ and $\omega_g = 2\pi i g(\tau) d\bar{\tau}$

We may compute $\langle reg_{\mathcal{D}}(\theta_{prod}), \omega_f \wedge \bar{\omega}_g \rangle$

$$= \frac{1}{2\pi i} \int_{X_0(p)} \log\left|\frac{\Delta(p\tau)}{\Delta(\tau)}\right| \omega_f \wedge \bar{\omega}_g$$

Recall $\eta(\tau) = q^{\frac{1}{12}} \prod_{n\geq 1}(1-q^n)$ and $\Delta = \eta^{24}$. can multiply $E_{2s,-s}(\tau,0)$ by $\Gamma(2s)$ to kill γ

Kronecker limit formula: $E_{2s,-s}(\tau,0) = \frac{1}{s-1} + 2\gamma - \log(4\pi) - \log(v|\eta(\tau)|^4) + O(s-1)$

where $\gamma = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx$ is the Euler's constant

$$\Rightarrow \quad E^{(p)}_{2s,-s}(z,s) = p^{-s} E_{2s,-s}(p\tau, s) - p^{-2s} E_{2s,-s}(\tau,s)$$
$$= \frac{p^{-1}-p^{-2}}{s-1} + \text{const} + \log\left(\left|\frac{\eta(p\tau)}{\eta(\tau)}\right|^4\right) + O(s-1)$$
$$= \frac{p^{-1}-p^{-2}}{s-1} + \text{const} + \frac{1}{6}\log\left(\left|\frac{\Delta(p\tau)}{\Delta(\tau)}\right|\right) + O(s-1)$$

Corollary: $\langle reg_{\mathcal{D}}(\theta_{prod}), \omega_f \wedge \bar{\omega}_g \rangle_{Pet}$

$$= \frac{1}{2\pi i}\left(\int_{X_0(p)} \left(6\, E^{(p)}_{2s,-s}(z,s)\omega_f \wedge \bar{\omega}_g\right) + \underbrace{\int_{X_0(p)} \left(\frac{\text{const}}{s-1} + \text{const}\right)\omega_f \wedge \bar{\omega}_g}_{=0 \text{ b/c } \langle f,g\rangle = 0.}\right)\Bigg|_{s=1}$$

$$\sim \frac{1}{2\pi i}\cdot \pi^{-3}\cdot L(f\times g,2)\cdot \underbrace{(2\pi i)^2}_{\text{from } f \text{ and } g}$$

$I(f,g,s)\sim\pi^{l-2k+1-2s}L(f\times g,k-1+s)$

$k=l=2,\ s=1$

$$\sim \frac{i}{\pi^2}L(f\times g,2)$$ maybe mult. by $(2\pi i)^2$?

§3 Beilinson's conjecture for $L(f\times g,2)$

$M:=M(f)\otimes M(g)$ Hodge type of $M(f\times g)$ is $(0,2),(1,1)\times 2,(2,0)$

$$H^2_{\mathcal{M}}(X_1(N),\mathbb{Q}(2))\otimes\mathbb{R}\longrightarrow \mathrm{Coker}\left(\alpha^{\mathbb{R}}_{M(2)}:M(2)^+_B\otimes\mathbb{R}\longrightarrow M(f\times g)_{dR}/F^2\otimes\mathbb{R}\right)$$

What is $\dfrac{M_{dR}\otimes\mathbb{R}}{F^2M_{dR}\otimes\mathbb{R}+M_B(2)^+\otimes\mathbb{R}}$? $M(f)_{dR}=\mathbb{Q}\omega_f\oplus\mathbb{Q}\eta_f$, $M(f)_{dR}\otimes\mathbb{R}=\mathbb{R}\omega_f\oplus\mathbb{R}\,\underbrace{\bar\omega_f}_{:=F_\infty(\omega_f)}$

Note: $\bar\omega_f$ is an $\mathbb{R}$-linear comb. of ω_f and η_f

Claim: $\omega_f\otimes\eta_g\notin F^2M_{dR}\otimes\mathbb{R}\oplus M(2)^+_B\otimes\mathbb{R}$, b/c $\omega_f\otimes M(g)_{dR}\notin M^+_B\otimes\mathbb{R}$

So $\omega_f\otimes\eta_g$ gives a basis of the quotient $M_{dR}\otimes\mathbb{R}/F^2M_{dR}\otimes\mathbb{R}\oplus M_B(2)^+\otimes\mathbb{R}$

But with the correct rational structure, we should use $(2\pi i)^{-3}\langle f,f\rangle^{-1}\omega_f\otimes\eta_g$

b/c write $\omega_f=\Omega_f^+e_f^++\Omega_f^-e_f^-$, $\eta_f=\Theta_f^+e_f^++\Theta_f^-e_f^-$ & $\Omega_f^+\Theta_f^--\Omega_f^-\Theta_f^+\sim(2\pi i)$

$$(\omega_f\otimes\eta_g)\wedge(\omega_f\otimes\omega_g)\wedge(2\pi i)^2(e_f^+\otimes e_g^+)\wedge(2\pi i)^2(e_f^-\otimes e_g^-)=\pm(2\pi i)^4\det\begin{pmatrix}\Omega_f^+\Theta_g^- & \Omega_f^+\Omega_g^-\\ \Omega_f^-\Theta_g^+ & \Omega_f^-\Omega_g^+\end{pmatrix}e_{++}\wedge e_{+-}\wedge e_{-+}\wedge e_{--}$$

$$\sim(2\pi i)^5\Omega_f^+\Omega_f^-\cdot e^{top}_{\pm\pm}\sim(2\pi i)\,\Omega_f^+\Omega_f^-\,e^{top}_{dR}\sim(2\pi i)^3\langle f,f\rangle e^{top}_{dR}$$

Consider the duality pairing $\dfrac{M_{dR}\otimes\mathbb{R}}{F^2M_{dR}\otimes\mathbb{R}\oplus M(2)^+_B\otimes\mathbb{R}}\otimes\underbrace{(F^1M_{dR})^-}_{\text{basis } \omega_f\otimes\bar\omega_g-\bar\omega_f\otimes\omega_g}\longrightarrow\mathbb{R}$

$$\text{So } \det\alpha^{\mathbb{R}}_{M(2)}=\frac{\langle \mathrm{reg}(\Theta_{prod}),\bar\omega_f\otimes\omega_g+\omega_f\otimes\bar\omega_g\rangle_{Pet}}{\langle(2\pi i)^{-3}\langle f,f\rangle^{-1}\omega_f\otimes\eta_g,\ \bar\omega_f\otimes\omega_g-\omega_f\otimes\bar\omega_g\rangle_{Pet}}\sim\frac{\frac{1}{\pi^2 i}L(f\times g,2)}{(2\pi i)^{-3}\langle f,f\rangle^{-1}\cdot\underbrace{\langle\eta_g,\omega_g\rangle_{Pet}}_{\in 2\pi i\mathbb{Q}^\times}\cdot\underbrace{\langle\omega_f,\bar\omega_f\rangle_{Pet}}_{=\langle f,f\rangle(2\pi i)^2}}$$

$$\sim L(f\times g,2)$$

§4 Beilinson conjecture for $H^2_{\mathcal{M}}(X, \mathbb{Q}(2))$

We need to work with $X_1(N)$ instead.

For a nontrivial Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to L^\times$, consider the Eisenstein series

$$E_{k+2s,-s,\chi}(\tau) := \pi^{-(k+s)} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\chi^{-1}(d)}{(c\tau+d)^k} \cdot \left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2} \right)^s \right) \qquad wt = k$$

<u>Fact</u>: We have differential operator $\delta_{k,-s} := (2\pi i)^{-1} \left(\frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar{\tau}} \right)$

$$\delta_{k,-s}\left(E_{k+s,-s,\chi}(\tau)\right) = (k+s) \cdot E_{k+2s,-s+1,\chi}(\tau) \leftarrow \text{of wt } k+2$$

<u>Fact</u>: When $s=0$, this is holomorphic Eisenstein series (but with a pole factor)

We write $E^{hol}_{k,\chi}(\tau) := \pi^{-k} \cdot \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\chi^{-1}(d)}{(c\tau+d)^k}$

<u>Fact</u>: $\exists\, u_\chi \in \mathcal{O}(Y_1(N))^\times \otimes L$ s.t. $\quad E_{2,-1,\chi}(\tau) = \log|u_\chi|$

Let ε be another character.

Consider $H^1_{\mathcal{M}}(Y, L(1)) \otimes H^1_{\mathcal{M}}(Y, L(1)) \xrightarrow{\cup} H^2_{\mathcal{M}}(Y, L(2)) = K_2^M(Y) \otimes L$

$$u_\chi \otimes u_\varepsilon \longmapsto u_\chi \otimes u_\varepsilon$$

where $K_2^M(Y) = \mathcal{O}(Y)^\times \otimes \mathcal{O}(Y)^\times / \langle a \otimes 1-a, \text{ if } a, 1-a \in \mathcal{O}(Y)^\times \rangle$ ← Steinberg relation

<u>Fact</u> There's an exact sequence: $0 \to K_2^M(X) \to K_2^M(Y) \xrightarrow{\partial} \mathcal{O}(C)^\times = \mathbb{Q}^\times \oplus \mathbb{Q}^\times$ (at C_0, at C_∞)

$$u_\chi \otimes u_\varepsilon \longmapsto 0$$

$$\begin{array}{ccc}
H^1_{\mathcal{M}}(Y, L(1)) \times H^1_{\mathcal{M}}(Y, L(1)) & \xrightarrow{\cup} & H^2_{\mathcal{M}}(Y, L(2)) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \mathrm{reg}_{\mathcal{D}} \\
H^1(Y, \mathbb{R}_{\mathcal{D}}(1)) \times H^1(Y, \mathbb{R}_{\mathcal{D}}(1)) & \xrightarrow{\cup} & H^2(Y, \mathbb{R}_{\mathcal{D}}(2))
\end{array}$$

$$\eta_\chi = \log|u_\chi| \quad \eta_\varepsilon = \log|u_\varepsilon| \quad \xrightarrow{\cup} \quad ?$$

$$\omega_\chi = \frac{du_\chi}{u_\chi} \qquad \omega_\varepsilon = \frac{du_\varepsilon}{u_\varepsilon}$$

We are taking the cup product of

$$\begin{bmatrix} \Omega_X^{\geq 1} \to A^\bullet_{X,\mathbb{C}} \\ \oplus \\ A_{X,\mathbb{R}(1)} \end{bmatrix} \otimes \begin{bmatrix} \Omega_X^{\geq 1} \to A^\bullet_{X,\mathbb{C}} \\ \oplus \\ A_{X,\mathbb{R}(1)} \end{bmatrix} \longrightarrow \begin{bmatrix} \Omega_X^{\geq 2} \to A^\bullet_{X,\mathbb{C}} \\ \oplus \\ A_{X,\mathbb{R}} \end{bmatrix} \qquad 1\otimes f_2 + g_1 \otimes 1$$

$$\underset{0}{(\omega_\chi\ ,\ \log|u_\chi|)} \quad \underset{0}{(\omega_\varepsilon\ ,\ \log|u_\varepsilon|)} \longmapsto \omega_\chi \wedge \omega_\varepsilon\ ,\ \log|u_\chi|(\omega_\varepsilon + \bar{\omega}_\varepsilon)$$

$$H^2_{\mathcal{D}}(X, \mathbb{R}_{\mathcal{D}}(2)) = H^1(X,\mathbb{R}) \big/ \underbrace{F^2 H^1_{dR}(X)}_{0}$$

So we can pair it with $H^0(X, \Omega^1_X)$, $f \in S_2(\Gamma_1(N), \psi)$ s.t. $\psi \bar{\varepsilon} \chi = \underline{1}$.

$$\langle \mathrm{reg}_{\mathcal{D}}(u_\chi \otimes u_\varepsilon), \omega_f \rangle = \int_{X_1(N)} \underbrace{\log|u_\chi|}_{E_{2,-1,\chi}(\tau)} \, d\log u_\varepsilon \wedge \omega_f .$$

$$\frac{d}{d\tau}\left(E_{2,-1,\chi}(\tau)\right) \cdot d\tau \sim E^{hol}_{0,\varepsilon}(\tau)(2\pi i\, d\tau)$$

a holomorphic Eisenstein series

By Rankin-Selberg $\quad = L(f \times E^{hol}_{2,\varepsilon}, 2) \cdot (2\pi i)^2 \cdot \pi^{-3}$ ← RS

from two $2\pi i\, d\tau$

But $\rho_{E^{alg}_{2,\varepsilon}} = \mathbb{Q} \oplus \mathbb{Q}(\varepsilon)(-1)$ $\quad \sim \underbrace{L(f\times\varepsilon, 1)}_{\Omega_f^\pm} \cdot L(f,2) \cdot \pi^{-1}$

<u>Interpretation</u> $\quad H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \longrightarrow \mathrm{Coker}\left(M(2)^+_B \otimes \mathbb{R} \to M_{dR}^{\otimes\mathbb{R}} / \underbrace{F^2 M_{dR}}_{0}\right) \qquad M = M(f)$

Consider duality pairing $\dfrac{M_{dR}\otimes\mathbb{R}}{M(2)^+_B \otimes \mathbb{R}} \otimes F^1 M_{dR} \longrightarrow \mathbb{R} \qquad \Lambda^2 M = \mathbb{Q}(1)$

$$(2\pi i)^{-1} e_- \ \times\ \underset{}{\omega_f} \longmapsto (2\pi i)^{-1}\Omega_f^+ \qquad \text{under } e^{top}_{+-}$$

$\omega_f = \Omega_f^+ e^+ + \Omega_f^- e^-$

$$\mathrm{reg}_{\mathcal{D}}(u_\chi \otimes u_\varepsilon) \times \omega_f \longmapsto \pi^{-1}\Omega_f^+ L(f,2)$$

So $\mathrm{reg}_{\mathcal{D}}(u_\chi \otimes u_\varepsilon) \sim L(f,2)$

§A. Siegel unit over modular curve.

Theorem. Let $D \in \mathbb{Z}$ be s.t. $(6, D) = 1$. There is one and only one rule ϑ_D to associates to each elliptic curve $E \to S$ a section $\vartheta_D^{(E/S)} \in \mathcal{O}(E - \ker[m_D])^\times$

(1) $\operatorname{div}\left(\vartheta_D^{(E/S)}\right) = D^2[e] - [\ker m_D]$

(2) For $S' \xrightarrow{g} S$, put $E' := E \times_S S'$, then $g^* \vartheta_D^{(E/S)} = \vartheta_D^{(E'/S')}$

(3) If $\alpha : E \to E'$ is an isogeny of degree prime to D, then $\alpha_* \vartheta_D^{(E/S)} = \vartheta_D^{(E'/S)}$

Explicitly, if $E = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, $\vartheta_D^{(E_\tau/\mathbb{C})}(z) = (-1)^{\frac{D-1}{2}} \Theta(z,\tau)^{D^2} \Theta(Dz,\tau)^{-1}$

$$\text{where } \Theta(z,\tau) = q^{\frac{1}{12}}\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) \prod_{n>0} (1 - q^n t)(1 - q^n t^{-1}) \qquad q = e^{2\pi i \tau},\ t = e^{2\pi i z}$$

Proof: Note that $D^2[e] - [\ker m_D]$ is a principal divisor.

Pick any function $\vartheta_D'^{(E/S)} \in \mathcal{O}(E)^\times$, s.t. $\operatorname{div}(\vartheta_D'^{(E/S)}) = D^2[e] - [\ker m_D]$.

Should use $\vartheta_D^{(E/S)} = \lambda \cdot \vartheta_D'^{(E/S)}$ for some $\lambda \in \mathcal{O}(S)^\times$.

Consider $m_3 : E \to E$, $x \mapsto 3x$, $m_{3*}(\vartheta_D'^{(E/S)})$ has divisor $m_{3*}(D^2[0] - [\ker m_D]) = D^2[0] - [\ker m_D]$

So $m_{3*}(\vartheta'^{(E/S)}) = u_3 \cdot \vartheta'^{(E/S)}$ for $u_3 \in \mathcal{O}(S)^\times$

and $m_{2*}(\vartheta'^{(E/S)}) = u_2 \cdot \vartheta'^{(E/S)}$ for $u_2 \in \mathcal{O}(S)^\times$

$$\left.\begin{array}{l} \text{Want } m_{3*}(\lambda \theta'^{(E/S)}) = \lambda \cdot \theta'^{(E/S)} \Rightarrow \lambda^8 \cdot u_3 = 1 \\ m_{2*}(\lambda \theta'^{(E/S)}) = \lambda \cdot \theta'^{(E/S)} \Rightarrow \lambda^3 \cdot u_2 = 1 \end{array}\right\} \lambda = \frac{u_3}{u_2^3} \Rightarrow \vartheta^{(E/S)} := \frac{\vartheta'^{(E/S),3}\, m_{3*}(\theta'^{(E/S)})}{\left(m_{2*}\vartheta'^{(E/S)}\right)^3}$$

Moreover, this definition does not depend on the initial choice of $\vartheta'^{(E/S)}$.

If $\alpha : E \to E'$ is an isogeny of degree prime to D

then $\alpha_* \vartheta'^{(E/S)}$ is a function on $\vartheta'^{(E'/S)}$ with divisor $D^2 \cdot [0] - [\ker m_D]$.

$$\text{So } \alpha_* \vartheta^{(E/S)} = \alpha_*\left(\frac{\theta'^{(E/S),3}\, m_{3*}(\theta'^{(E/S)})}{m_{2*}(\theta'^{(E/S)})^3}\right) = \frac{\theta'^{(E'/S),3}\, m_{3*}(\theta'^{(E'/S)})}{m_{2*}\theta'^{(E'/S)3}} = \vartheta^{(E'/S)}$$

Explicit formula : Θ is not a function on E_τ ; $\Theta(z+1,\tau) = \Theta(z,\tau)$

But $\Theta(z+\tau,\tau) = \Theta(tq,q) = q^{\frac{1}{12}}\left(t^{\frac{1}{2}}q^{\frac{1}{2}} - t^{-\frac{1}{2}}q^{-\frac{1}{2}}\right)\prod_{n>0}(1-q^{n+1}t)(1-q^{n-1}t^{-1})$

$\qquad = -t^{-1}q^{-\frac{1}{2}}\Theta(t,q)$

Still $\vartheta_D^{(E_\tau/\mathbb{C})}(z) = (-1)^{\frac{D-1}{2}}\Theta(z,\tau)^{D^2}\Theta(Dz,\tau)^{-1}$ is $\mathbb{Z}\oplus\mathbb{Z}\tau$-invariant

(Note: $\Theta(Dz+D\tau,\tau) = (-1)^D\left(t^{-D}q^{-\frac{1}{2}}\right)\left(t^{-D}q^{-1}\cdot q^{-\frac{1}{2}}\right)\cdots\left(t^{-D}q^{-(D-1)}q^{-\frac{1}{2}}\right) = (-1)^D t^{-D^2} q^{-\frac{D^2}{2}}$.)

Also, Θ is $SL_2(\mathbb{Z})$-invariant by classical theory of Jacobi forms.

Thus the expression for $\vartheta_D(q,t)$ almost satisfies (3), up to a unit

$$\vartheta_D(q,t) = (-1)^{D-1} q^{\frac{D^2-1}{12}} t^{-\frac{D(D-1)}{2}} \prod_{n\geq 0}\frac{(1-q^n t)^{D^2}}{1-q^n t^D}\prod_{n>0}\frac{(1-q^n t^{-1})^{D^2}}{1-q^n t^{-D}} \in \mathbb{Z}\left[t,\frac{1}{t(1-t^D)}\right][[q]]$$

$\vartheta_D(q,t)$ is a unit in $\mathbb{Z}\left[t,\frac{1}{t(1-t^D)}\right][[q]]^\times$, but q-expansion principle,

$\vartheta_D(q,t)$ almost satisfies (3) up to ± 1. then check $/\mathbb{R}$. $\square$

Now, consider $X = X_1(N)$. Take D s.t. $(N,D)=1$

$$\begin{tikzcd} \mathcal{E}^{sm} \arrow[d] \\ X \arrow[u, bend right, "i_N"'] \end{tikzcd} \qquad i_N : \underline{\mathbb{Z}/N\mathbb{Z}} \hookrightarrow \underbrace{\mathcal{E}[N]}$$

(changed from earlier of this lecture series)

$$\underline{\mathbb{Z}/N\mathbb{Z}} \supseteq 1_S \xrightarrow{i'_N} \mathcal{E}[N]$$

away from $\mathcal{E}[D]$

Define $\tilde{\theta} := i_N'^{-1}\left(\vartheta_D^{(\mathcal{E}/Y)}\right) \in \mathcal{O}(Y)^\times = H^1_{\mathcal{M}}(Y,\mathbb{Z}(1))$