Special values of L-functions 3

—— p-adic Dirichlet L-functions

<u>Recall</u>: For any $f \in C^\infty(\mathbb{Z}_p, \mathbb{Z}_p)$, it admits a <u>Mahler expansion</u>

$$f(x) = \sum_{n \geq 0} a_n(f)\binom{x}{n} \quad \text{for } a_n \in \mathbb{Q}_p, \ a_n \to 0 \text{ as } n \to +\infty$$

There is an isomorphism (with a pretty formula) called <u>Amice transform</u>

$$\begin{aligned} \mathrm{Hom}_{\mathrm{cont}}(C^0(\mathbb{Z}_p,\mathbb{Z}_p),\mathbb{Z}_p) \cong \mathcal{D}_0(\mathbb{Z}_p,\mathbb{Z}_p) &\xrightarrow{\sim} \mathbb{Z}_p[\![T]\!] \\ \mu &\longmapsto A_\mu(T) = \int_{\mathbb{Z}_p} (1+T)^x \, d\mu(x) \end{aligned}$$

§1. An intrinsic construction of the Amice transform.

<u>Recall</u>: For a profinite group G, we have an <u>Iwasawa algebra</u>

$$(\Lambda(G) =)\, \mathbb{Z}_p[\![G]\!] \cong \varprojlim_{\substack{H \trianglelefteq G \\ \text{finite}}} \mathbb{Z}_p[G/H]$$

<u>Example</u>: $\mathbb{Z}_p[\![\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![T]\!]$

$$[1] - 1 \longleftarrow T$$

<u>Lemma/Notation</u> If $X = \varprojlim_i X_i$ is a profinite set (assume in the inverse limit, each X_i is finite and $X_j \twoheadrightarrow X_i$ $(j > i)$.)

Then the space of <u>p-adic measures</u> on X is given by

Proof: $$\begin{aligned} \mathcal{D}_0(X,\mathbb{Z}_p) &= \varprojlim_i \mathcal{D}_0(X_i,\mathbb{Z}_p) \\ &= \varprojlim_i \mathrm{Hom}_{\mathrm{cont}}(C^0(X_i,\mathbb{Z}_p),\mathbb{Z}_p) \\ &= \varprojlim_i \mathbb{Z}_p[X_i] \leftarrow \text{this is to give each point of } X_i \text{ a weight} \\ &=: \mathbb{Z}_p[\![X]\!]. \end{aligned}$$

In the special case when $X = G$ is a profinite group, we get a canonical isomorphism

$$\mathcal{D}_0(G,\mathbb{Z}_p) \cong \mathbb{Z}_p[\![G]\!].$$ ← Iwasawa algebra.

<u>Lemma</u> The multiplication for the ring structure in $\mathbb{Z}_p[\![G]\!]$ corresponds to the convolution product on $\mathcal{D}_0(G,\mathbb{Z}_p)$:
$$\int_G f(g)\, d(\mu_1 * \mu_2)(g) := \int_G \int_G f(gh)\, d\mu_1(g)\, d\mu_2(h)$$

Proof: By taking limit, it is enough to prove this when G is finite.

In this case, we may view μ_i as a function in $\mathbb{Z}_p[G]$

$$\langle f, \mu_1 \cdot \mu_2 \rangle = \sum_{g\in G} f(g)\, (\mu_1 \cdot \mu_2)(g) = \sum_{g\in G} f(g) \sum_{h\in G} \mu_1(h)\, \mu_2(h^{-1}g)$$

$$\overset{h_1 = h^{-1}g}{=} \sum_{h\in G} \sum_{h_1\in G} \mu_1(h)\, \mu_2(h_1) \cdot f(hh_1). \qquad \square$$

<u>Theorem</u> The Amice transform $\mu \longmapsto A_\mu(T) = \int_{\mathbb{Z}_p} (1+T)^x\, d\mu(x)$ is exactly the composition of the following canonical isomorphisms
$$\mathcal{D}_0(\mathbb{Z}_p,\mathbb{Z}_p) \cong \mathbb{Z}_p[\![\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!].$$

Proof:
$$\mathcal{D}_0(\mathbb{Z}_p) = \varprojlim_{m\to\infty} \mathcal{D}_0(\mathbb{Z}_p/p^m\mathbb{Z}_p) \cong \varprojlim_m \mathbb{Z}_p[\mathbb{Z}_p/p^m\mathbb{Z}_p] \cong \varprojlim_m \mathbb{Z}_p[T]/(1+T)^{p^m}-1$$

$$\mu \longmapsto \lim_{m\to+\infty} \sum_{a\in \mathbb{Z}_p/p^m\mathbb{Z}_p} \mu(a+p^m\mathbb{Z}_p)\cdot [a] \longmapsto \lim_{m\to+\infty} \sum_{a\in \mathbb{Z}_p/p^m\mathbb{Z}_p} \mu(a+p^m\mathbb{Z}_p)\,(1+T)^a$$

$$\| $$

$$\int_{\mathbb{Z}_p} (1+T)^x\, d\mu(x) = A_\mu(T).$$

This is indeed a proof....

§2. <u>Furthur properties of measures on $\mathbb{Z}_p$</u>

<u>Operations on $\mathbb{Z}_p[\![T]\!] \cong \mathbb{Z}_p[\![\mathbb{Z}_p]\!]$</u>

<u>Rmk</u>: $G \rightsquigarrow \mathbb{Z}_p[\![G]\!]$ is functorial in G.

i.e. if $G \to H$ a cont. homo. $\Rightarrow \mathbb{Z}_p[\![G]\!] \to \mathbb{Z}_p[\![H]\!]$ cont. homo.

① φ-action: $\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p$ group homomorphism

$\Rightarrow$ ring homomorphism $\varphi : \mathbb{Z}_p[[T]] \longrightarrow \mathbb{Z}_p[[T]]$

$[1] = 1+T \longmapsto [p] = (1+T)^p$, so $\varphi(T) = (1+T)^p - 1$.

② ψ-action: $\mathbb{Z}_p = \coprod_{i=0}^{p-1} (i + p\mathbb{Z}_p)$;

so $\mathbb{Z}_p[[T]] = \bigoplus_{i=0}^{p-1} (1+T)^i \varphi(\mathbb{Z}_p[[T]])$ b/c cosets form a basis

Define the ψ-operator $\psi : \mathbb{Z}_p[[T]] \longrightarrow \mathbb{Z}_p[[T]]$ as follows:

Write h (uniquely) as $\sum_{i=0}^{p-1} (1+T)^i \varphi(h_i)$. Define $\psi(h) = h_0$

$\Rightarrow \psi \circ \varphi(h) = h$.

③ Γ-action (will not be used for p-adic L-functions)

$\Gamma \simeq \mathbb{Z}_p^\times$; for $a \in \mathbb{Z}_p^\times$, write γ_a for the corresponding element.

$\rightsquigarrow$ $\text{mult}_a : \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p$ induces a ring homomorphism

$\gamma_a : \mathbb{Z}_p[[T]] \longrightarrow \mathbb{Z}_p[[T]]$ $\qquad \gamma_a : 1+T = [1] \longmapsto [a] = (1+T)^a$

$\gamma_a(T) = (1+T)^a - 1$.

Γ-action commutes with φ and ψ-actions.

Interpretation in terms of measures on $\mathbb{Z}_p$

① φ-action

$$\begin{array}{ccc} \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p) & \xrightarrow{\sim} & \mathbb{Z}_p[[T]] \\ \downarrow \varphi & & \downarrow \varphi \\ \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p) & \xrightarrow{\sim} & \mathbb{Z}_p[[T]] \end{array}$$

$$\int (1+T)^x \, d\varphi(\mu)(x) = \varphi\left(\int (1+T)^x d\mu(x)\right) = \int (1+T)^{px} d\mu(x)$$

So $\varphi(\mu)$ is defined to be $\int f(x)\, d\varphi(\mu)(x) := \int_{\mathbb{Z}_p} f(px)\, d\mu(x)$

② Γ-action For $\gamma_a \in \Gamma$, $\int (1+T)^x d\gamma_a(\mu)(x) = \gamma_a\left(\int (1+T)^x d\mu(x)\right) = \int (1+T)^{ax} d\mu(x)$

So $\gamma_a(\mu)$ is defined to be $\int f(x)\, d\gamma_a(\mu)(x) := \int_{\mathbb{Z}_p} f(ax)\, d\mu(x)$

i.e. the induced action of $\mathbb{Z}_p^\times$ on measures of $\mathbb{Z}_p$.

③ <u>shift $x \mapsto x+a$</u> For $a \in \mathbb{Z}_p$, define $s_a : \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p) \longrightarrow \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p)$

$$\int f(x)\, d s_a(\mu)(x) := \int f(x+a)\, d\mu(x)$$

So $A_{s_a(\mu)}(T) = \int (1+T)^x d s_a(\mu)(x) = \int (1+T)^{x+a} d\mu(x) = (1+T)^a \cdot A_\mu(T)$

④ For each $h \in \mathbb{Z}_p[[T]]$, write $h = \sum_{i=0}^{p-1} (1+T)^i \varphi(h_i)$ $\qquad h \leftrightarrow \mu,\ h_i \leftrightarrow \mu_i$

$$\int_{\mathbb{Z}_p} f(x)\, d\mu(x) = \sum_{i=0}^{p-1} \int_{\mathbb{Z}_p} f(x)\, d(s_a \varphi(\mu_i))(x) = \sum_{i=0}^{p-1} \int_{\mathbb{Z}_p} f(i+px)\, d\mu_i(x)$$

<u>Cor 1.</u>

$$\underbrace{\int_{\mathbb{Z}_p} g(x)\, d\psi(\mu)(x)}_{f(px)} = \int_{\mathbb{Z}_p} \begin{cases} g\left(\frac{x}{p}\right) & \text{if } p \mid x \\ 0 & \text{if } p \nmid x \end{cases} d\mu(x)$$

i.e. to first restrict $d\mu$ to $p\mathbb{Z}_p$ and view as a measure on $\mathbb{Z}_p$ via $p\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p$.

<u>Cor 2.</u> For each i, $\mathrm{Res}_{i+p\mathbb{Z}_p}(d\mu) \longleftrightarrow (1+T)^i \varphi(h_i)$

More generally, for $r \in \mathbb{Z}_{\geq 1}$, write $h = \sum_{i=0}^{p^r-1} (1+T)^i \varphi^r(h_i)$,

we have $\mathrm{Res}_{i+p^r\mathbb{Z}_p}(d\mu_h) \longleftrightarrow (1+T)^i \varphi^r(h_i)$.

<u>Definition</u>: For $\mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p)$, $\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu) := (1 - \varphi\psi)(\mu)$ is the measure given by

Write $A_\mu(T) = \sum_{i=0}^{p-1} (1+T)^i \varphi(h_i)$, then

$$A_{\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu)}(T) = \sum_{i=1}^{p-1} (1+T)^i \varphi(h_i) = A_\mu(T) - \varphi(h_0) = (1 - \varphi\psi)(A_\mu)$$

$\square$

<u>Cor.</u> $\mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p)$ is supported on $\mathbb{Z}_p^\times$ iff $\psi(\mu) = 0$.

Proof: $\mu = \mathrm{Res}_{\mathbb{Z}_p^\times}(\mu) \Longleftrightarrow \mu = \mu - \varphi\psi(\mu) \Longleftrightarrow \psi(\mu) = 0$. $\square$

§3 p-adic Dirichlet L-functions

Target Theorem Let $\eta \neq \mathbb{1}$ be a Dirichlet character of conductor prime-to-p.

Then there exists a p-adic measure μ_η on $\mathbb{Z}_p^\times \simeq \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ with values in $\mathbb{Q}_p(\eta)$

s.t. for any primitive finite character $\eta_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$ and any $n \in \mathbb{Z}_{\geq 0}$,

$$\int_{\mathbb{Z}_p^\times} \eta_p(x)\, x^n \, d\mu_\eta(x) = L^{(p)}(\eta\eta_p, -n). \qquad (*)$$

Write $\mathcal{O}$ for ring of integers in $\mathbb{Q}_p(\eta)$.

Lemma. If $\mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathcal{O})$ supported on $\mathbb{Z}_p^\times$, with Amice transform $A_\mu(T)$

then
$$\int_{\mathbb{Z}_p^\times} x^n \, d\mu(x) = \int_{\mathbb{Z}_p} x^n \, d\mu(x) = \left((1+T)\frac{d}{dT}\right)^n (A_\mu(T))\Big|_{T=0}$$

(can get a formula for $\int_{\mathbb{Z}_p^\times} \eta_p(x) x^n d\mu(x)$ but it's less pretty.)

Proof: By Amice transform $\int_{\mathbb{Z}_p} (1+T)^x d\mu(x) = A_\mu(T)$.

Apply $(1+T)\frac{d}{dT}$ to this equation $\Rightarrow$ $\int_{\mathbb{Z}_p} x\cdot(1+T)^x d\mu(x) = (1+T)\frac{d}{dT} A_\mu(T)$

Applying this n times $\Rightarrow$ $\int_{\mathbb{Z}_p} x^n (1+T)^x d\mu(x) = \left((1+T)\frac{d}{dT}\right)^n A_\mu(T)$

Setting $T=0$ ✓

Recall: For η a Dirichlet character of conductor N (prime-to-p),

$$f_\eta(t) := \frac{\sum_{a=1}^{N-1} \eta(a)\cdot e^{-at}}{1-e^{-Nt}} \rightsquigarrow L(\eta,-n) = (-1)^n f_\eta^{(n)}(0) = \left(\frac{-d}{dt}\right)^n (f_\eta)\Big|_{t=0}$$

Put
$$f_\eta^{(p)}(t) := \frac{\sum_{\substack{a=1 \\ (a,pN)=1}}^{pN-1} \eta(a) e^{-at}}{1-e^{pNt}} \rightsquigarrow \left(-\frac{d}{dt}\right)^n (f_\eta^{(p)})\Big|_{t=0} = L^{(p)}(\eta,-n)$$

$$\| $$

$$L(\eta,-n)\,(1-\eta(p)p^n)$$

Note: $(1+T)\frac{d}{dT} = \frac{d}{d\log(1+T)}$, So if we set $1+T = e^{t}$, then $(1+T)\frac{d}{dT} = -\frac{d}{dt}$

(and $T=0 \longleftrightarrow t=0$)

For $\displaystyle A_\eta^{\mathbb{Z}_p}(T) := \frac{\sum_{a=1}^{N-1} \eta(a)(1+T)^a}{1-(1+T)^N} \in \mathcal{O}[[T]]$, $\displaystyle A_\eta(T) := \frac{\sum_{\substack{1\le a\le pN-1 \\ (a,pN)=1}} \eta(a)(1+T)^a}{1-(1+T)^{pN}}$

Need $p \nmid N$

$$\Rightarrow A_\eta^{\mathbb{Z}_p}(e^t) = f_\eta(t) \,,\quad A_\eta(e^t) = f_\eta^{(p)}(t)$$

So $\displaystyle \left(\left((1+T)\frac{d}{dT}\right)^n (A_\eta^{\mathbb{Z}_p})\right)\Big|_{T=0} = \left(-\frac{d}{dt}\right)^n (f_\eta)\Big|_{t=0.}$ and $\displaystyle \left((1+T)\frac{d}{dT}\right)^n (A_\eta)\Big|_{T=0} = \left(-\frac{d}{dt}\right)^n (f_\eta^{(p)})\Big|_{t=0}$

<u>Theorem</u> $A_\eta(T) := (1-\varphi\psi)(A_\eta^{\mathbb{Z}_p}(T))$ is the p-adic Dirichlet L-function,
i.e. satisfies $(*)$

Proof: We may formally write

$$A_\eta^{\mathbb{Z}_p}(T) = \frac{\sum_{a=1}^{N-1} \eta(a)(1+T)^a}{1-(1+T)^N} = \sum_{\substack{a\ge 1 \\ (a,N)=1}} \eta(a)(1+T)^a$$

$$\Rightarrow (1-\varphi\psi)(A_\eta^{\mathbb{Z}_p}(T)) = \sum_{\substack{a\ge 1 \\ (a,pN)=1}} \eta(a)(1+T)^a = \frac{\sum_{\substack{a=1 \\ (a,pN)=1}}^{pN-1} \eta(a)(1+T)^a}{1-(1+T)^{pN}} = A_\eta(T) \qquad \square$$

<u>Theorem</u> $A_\eta(T) \in \mathcal{O}[[T]]$ corresponds to a p-adic measure $\mu_\eta \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathcal{O})$

It satisfies the condition in the above theorem, i.e. μ_η is the p-adic L-function for η.

Proof: When η_p is trivial, this was essentially just proved:

$$L^{(p)}(\eta,-n) = \left(\left(-\frac{d}{dt}\right)^n f_\eta^{(p)}\right)\Big|_{t=0} = \left(\left((1+T)\frac{d}{dT}\right)^n A_\eta\right)\Big|_{T=0} = \int_{\mathbb{Z}_p^\times} x^n \, d\mu_\eta \,.$$

When $\eta_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ is nontrivial, the function corresponding to $L(\eta\cdot\eta_p, s)$ is

$$L(\eta\cdot\eta_p, s) \longleftrightarrow f_{\eta\eta_p}(t) := \frac{\sum_{n=1}^{p^rN-1} \eta\eta_p(n) e^{-nt}}{1-e^{-Np^rt}} \overset{\text{formally}}{=\!=} \sum_{\substack{n\ge 1 \\ (n,pN)=1}} \eta\eta_p(n) e^{-nt}$$

so that $\displaystyle L(\eta\cdot\eta_p, -n) = \left(-\frac{d}{dt}\right)^n (f_{\eta\cdot\eta_p})\Big|_{t=0} = \left((1+T)\frac{d}{dT}\right)^n (A_{\eta\eta_p}(T))\Big|_{T=0}$

for $A_{\eta\eta_p}(T) = \dfrac{\sum_{n=1}^{p^rN-1} \eta\eta_p(n)(1+T)^n}{1-(1+T)^{p^rN}} \underset{\text{formally}}{=} \sum_{\substack{n\geq 1 \\ (n,pN)=1}} \eta\eta_p(n)(1+T)^n$

$$= \sum_{\substack{m=1 \\ p\nmid m}}^{p^r-1} \eta_p(m)\, \mathrm{Res}_{m+p^r\mathbb{Z}_p}(A_\eta(T)) \qquad \square$$