Special values of L-functions 4

—— Class number formula and statement of Iwasawa Main Conjecture

§1. L-functions associated general Galois representations

Definition Let F be a number field. Let $M_F := \{\text{all places of } F\} \supseteq M_{F,f} = \{\text{all finite places}\}$

For each $v \in M_{F,f}$, write $q_v :=$ # of elements in the residue field at v.

Fix a prime p and embeddings $\mathbb{Q}^{alg} \hookrightarrow \bar{\mathbb{Q}}_p$, where $\mathbb{Q}^{alg} \subseteq \mathbb{C}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

A representation $\rho : \mathrm{Gal}_F \longrightarrow GL_n(\bar{\mathbb{Q}}_p) = GL(V)$ is called "nice" if

(1) ρ is ramified at only finitely many places $S \subseteq M_F$

(WLOG, S contains all p-adic places and all archimedean places)

(sometimes, we write $G_{F,S}$ for the Galois group $\mathrm{Gal}(F^S/F)$ for F^S the maximal extension of F that is unramified outside S. Then we have $\rho : G_{F,S} \to GL_n(\bar{\mathbb{Q}}_p)$.)

(2) For every place $v \in M_{F,f}$ that is not p-adic, let ϕ_v be a geometric Frobenius at v the charateristic polynomial of $\rho(\phi_v)$ on V^{I_v} belongs to $\mathbb{Q}^{alg}[x]$.

(3) For a p-adic place v of F, $\rho_v := \rho|_{\mathrm{Gal}_{F_v}}$ is de Rham and $\rho(\phi_v)$ on $\underline{D_{pst}(\rho_v)}^{I_{F_v}}$ has charateristic polynomial in $\mathbb{Q}^{alg}[x]$

↳ some p-adic Hodge theory construction.

Then for each $v \in M_{F,f}$, define the local L-factor

$$L_v(\rho_v, s) := \begin{cases} \dfrac{1}{\det(1-\rho_v(\phi_v) q_v^{-s}; V^{I_v})} & \text{if } v \text{ is not } p\text{-adic.} \\ \dfrac{1}{\det(1-\phi_v \cdot q_v^{-s}; D_{pst}(\rho_v)^{I_v})} & \text{if } v \text{ is } p\text{-adic.} \end{cases}$$

Put $L(\rho, s) := \prod_{v \in M_{F,f}} L_v(\rho_v, s)$ if it converges for

Rmk: One expects meromorphic continuation + functional equation $L(\rho, s) \leftrightarrow L(\rho^\vee, 1-s)$

But this is a very difficult question ("Solution": only look at those assoc. to autom. rep'ns.)

<u>Rmk</u>: When $\mathrm{Im}\,\rho$ is finite, we can ρ an <u>Artin representation</u>. The p-adic Hodge theory construction can be ignored.

Basic examples/properties

① Comparison with primitive Dirichlet character $\tilde{\eta}: \mathrm{Gal}_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\eta} \mathbb{C}^\times$

$$\text{geom Frob. } \phi_p \longmapsto p^{-1} \longmapsto \eta(p)^{-1}$$

So $L(\tilde{\eta}, s) = \prod_p \frac{1}{1-\eta(p)^{-1}p^{-s}} = L(\eta^{-1}, s)$ we will see that this is consistent

<u>Rmk</u>: L-function for $\tilde{\eta}$ automatically gives the premitive Dirichlet L-function.

② p-adic cyclotomic character $\chi_{cycl}: \mathrm{Gal}_F \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \hookrightarrow \mathbb{Z}_p^\times$

$$\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cycl}(\sigma)}$$

$$\begin{tikzcd}
\mathbb{Q}(\mu_{p^\infty}) \arrow[r, dash] \arrow[d, dash] & F(\mu_{p^\infty}) \arrow[d, dash] \\
\mathbb{Q} \arrow[r, dash] & F
\end{tikzcd}$$

In particular, for $v \nmid p$ $\chi_{cycl}(\phi_v) = q_v^{-1}$

Sometimes abbreviate to $\mathbb{Z}_p(1)$ or $\mathbb{Q}_p(1)$

Put $\mathbb{Z}_p(n) = \mathbb{Z}_p(1)^{\otimes n}$ for $n \geq 0$ and $\mathbb{Z}_p(-n) = \mathrm{Hom}(\mathbb{Z}_p(n), \mathbb{Z}_p)$

For a representation V of Gal_F as above, define $V(n) = V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$

$$L_v(V(n)_v, s) = \frac{1}{\det(\mathbb{1} - \rho(n)(\phi_v)\cdot q_v^{-s})} = \frac{1}{\det(\mathbb{1} - \rho(\phi_v) q_v^{-n-s})} = L_v(V_v, s+n)$$

$$\Rightarrow L(V(n), s) = L(V, n+s)$$

↖ The same works for p-adic places as well b/c ϕ_v action on $D_{pst}(V_v(n))$ is $q_v^{-n}\phi_v$-action on $D_{pst}(V_v)$

<u>Reinterpretation of p-adic Dirichlet L-function</u>:

Let N be an integer $p \nmid N$. Fix an embedding $\mathbb{Q}^{alg} \hookrightarrow \overline{\mathbb{Q}}_p$.

Let $\tilde{\eta}: \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{Q}^{alg,\times}$ be a nontrivial character

There exists a p-adic measure $\mu_{\tilde{\eta}}$ on $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$ ← max'l p-abelian extension of $\mathbb{Q}$

s.t. for any finite character $\tilde{\eta}_p: \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}_p^\times$, and any $n \in \mathbb{Z}_{\geq 0}$

if we form a p-adic rep'n $\langle \tilde{\eta}_p, -n \rangle : \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times \longrightarrow \bar{\mathbb{Q}}_p^\times$

$$x \longmapsto \tilde{\eta}_p(x)\, x^{-n}$$

then $\displaystyle\int_{\mathbb{Z}_p^\times} \langle \tilde{\eta}_p, -n \rangle (x)\, d\mu_{\tilde{\eta}}(x) = L(\tilde{\eta}\tilde{\eta}_p, -n) = L(\eta^{-1}\eta_p^{-1}, -n)$

This $d\mu_{\tilde{\eta}} = \iota^* d\mu_{\eta^{-1}}$ where $\iota : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ is $x \mapsto x^{-1}$.

③ If $\rho = \rho_1 \oplus \rho_2$, $L(\rho, s) = L(\rho_1, s) \cdot L(\rho_2, s)$

④ $\rho \rightsquigarrow$ a representation $\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \rho$ of $\mathrm{Gal}_{\mathbb{Q}}$, then

$$L_F(\rho, s) = L_{\mathbb{Q}}(\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \rho)$$

<u>Example</u>: Consider the trivial rep'n $\mathbb{1}_F : \mathrm{Gal}_F \to \mathbb{C}^\times$, the associated L-function is called the <u>Dedekind zeta function</u> $\displaystyle\zeta_F(s) = L_F(\mathbb{1}, s) = \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_F \\ \text{prime}}} \frac{1}{1 - \|\mathfrak{p}\|^{-s}} = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_F \\ \text{ideal}}} \frac{1}{\|\mathfrak{a}\|^s}$ $\quad \mathrm{Re}(s) > 1$

$$= L_{\mathbb{Q}}(\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{1}, s)$$

<u>Special case</u> $F = \mathbb{Q}(\zeta_N)$, $\quad \displaystyle\mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{1} = \bigoplus_{\text{character}} \eta$

$\eta : \mathrm{Gal}(F/\mathbb{Q}) \to \mathbb{C}^\times \longleftrightarrow$ primitive Dirichlet char of conductor $M \mid N$

So $\displaystyle\zeta_F(s) = \prod_{\substack{\text{characters} \\ \eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times}} L(\eta, s).$

Next, we will study the arithmetic properties of special values of $L(\eta, 0)$.

§2 Analytic class number formula.

<u>Functional equation for Dedekind ζ-function $\zeta_F(s)$</u>

Assume that F has r_1 real embeddings $\tau_1, \cdots \tau_{r_1}$ and r_2 pairs of complex embeddings $\tau_{r_1+i}, \bar{\tau}_{r_1+i}$

<u>Recall</u>: $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

Put $\Lambda_F(s) := \Gamma_{\mathbb{R}}(s)^{r_1} \cdot \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_F(s)$

Then $\Lambda_F(s) = |\Delta_F|^{\frac{1}{2}-s} \Lambda_F(1-s)$, where Δ_F is the discreminant of $F/\mathbb{Q}$

<u>Theorem</u>. (analytic class number formula)

If F is a number field, then the Dedekind zeta function $\zeta_F(s)$ has a simple pole at $s=1$,

$$\lim_{s\to 1} (s-1)\zeta_F(s) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot \mathrm{Reg}_F \cdot h_F}{w_F \cdot \sqrt{|D_F|}}$$

← This is $\mathrm{Vol}(\mathbb{A}_F^{\times,1}/F^\times)$ for the self-dual measure used in Tate's thesis $\qquad \mathbb{A}_F^{\times,1} = \{x \in \mathbb{A}_F^\times, |x|=1\}$

where D_F = discriminant of F, $h_F = \# Cl(\mathcal{O}_F)$, $w_F := \#\{\text{roots of unity in } F\}$ $\overset{\text{\tiny ||}}{\mu(F)}$

$$\mathrm{Reg}_F = \underset{\text{more carefully later}}{\text{volume}} \text{ of } \frac{\mathbb{R}^{r_1+r_2-1}}{\mathrm{reg}_F(\mathcal{O}_F^\times)} ,$$

Here, $\mathrm{reg}_F : \mathcal{O}_F^\times \longrightarrow (\mathbb{R}^{r_1+r_2})^{\text{sum}=0}$

$$u \longmapsto \left(c_i \log|\tau_i(u)|\right)_{i=1,\dots r_1+r_2} \quad \text{for } c_i = \begin{cases} 1 & \text{if } \tau_i \text{ is real} \\ 2 & \text{if } \tau_i \text{ is complex.} \end{cases}$$

explicitly, if $u_1, \dots, u_{r_1+r_2-1}$ generate $\mathcal{O}_F^\times/\mu(F)$

$$\mathrm{Reg}_F = \left| \det\left(c_i \log|\tau_i(u_j)| \right)_{i,j=1,\dots r_1+r_2-1} \right|$$

<u>A better formulation?</u> $\displaystyle\lim_{s\to 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\frac{h_F \cdot \mathrm{Reg}_F}{w_F}$

Proof by functional equation (as $s \to 0$)

$$\zeta_F(s) \cdot \Big(\underbrace{\pi^{-\frac{s}{2}} \Gamma\big(\tfrac{s}{2}\big)}_{\substack{\wr\wr \\ (\frac{2}{s})^{r_1}}}\Big)^{r_1} \cdot \Big(\underbrace{2(2\pi)^{-s} \Gamma(s)}_{\substack{\wr\wr \\ (\frac{2}{s})^{r_2}}}\Big)^{r_2} = \underbrace{|\Delta_F|^{\frac{1}{2}-s}}_{\substack{\downarrow \\ \sqrt{|\Delta_F|}}} \cdot \underbrace{\zeta_F(1-s)}_{\substack{\| \\ \left(-\frac{1}{s}\right) \cdot \frac{2^{r_1} (2\pi)^{r_2} \mathrm{Reg}_F \cdot h_F}{w_F \cdot \sqrt{|\Delta_F|}}}} \cdot \Big(\underbrace{\pi^{-\frac{1-s}{2}} \Gamma\big(\tfrac{1-s}{2}\big)}_{\substack{\wr\wr \\ (\pi^{-\frac{1}{2}} \cdot \sqrt{\pi})^{r_1}}}\Big)^{r_1} \cdot \Big(\underbrace{2(2\pi)^{s-1} \Gamma(1-s)}_{\substack{\wr\wr \\ (\frac{2}{2\pi})^{r_2}}}\Big)^{r_2}$$

$\square$

§3 Proof of analytic class number formula (nothing deep, but a very cute proof.)

Suffices to consider $s\in\mathbb{R}$ and $s\to 1^+$.

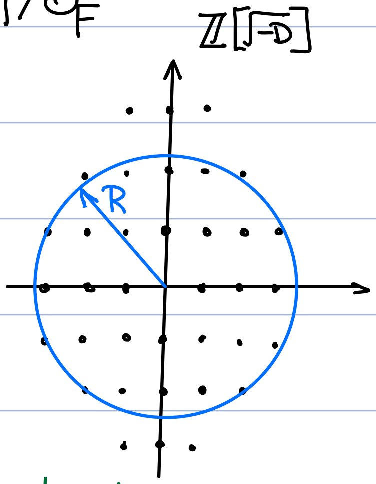
Case of $F=\mathbb{Q}$, $\zeta_{\mathbb{Q}}(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s}=\int_1^{\infty}\frac{1}{x^s}dx+O(1)=\frac{1}{1-s}x^{1-s}\Big|_1^{+\infty}+O(1)=\frac{1}{s-1}+O(1).$

Will only prove this when F is a quadratic field

$$\zeta_F(s)=\sum_{\substack{(0)\neq I\subseteq O_F\\ \text{ideal}}}\frac{1}{\|I\|^s}=\sum_{\substack{c\in Cl(O_F)\\ \text{ideal class}}}\sum_{I\in[c]}\frac{1}{\|I\|^s}$$

① When F is imaganary quadratic,

First compute $[c]=$ principal ideal class. $\longleftrightarrow\{\text{elements in } O_F\setminus\{0\}\}/O_F^\times$



$$\Rightarrow\sum_{\substack{\text{principal}\\ \text{ideals } I\neq 0}}\frac{1}{\|I\|^s}=\frac{1}{|O_F^\times|}\cdot\sum_{a\in O_F\setminus\{0\}}\frac{1}{(Na)^s}$$

$$\overset{Na\sim R^2}{=}\frac{1}{w_F}\int_{R=1}^{+\infty}\frac{1}{\underbrace{\frac{1}{2}\sqrt{|\Delta_F|}}}\left(2\pi R+O(1)\right)\frac{1}{R^{2s}}\,dR$$

↑density of lattice point e.g. $O_F=\mathbb{Z}[\sqrt{-D}]$

$\Delta_F=-4D$ vs. density $=\frac{1}{\sqrt{D}}$

$$=\frac{2}{w_F\sqrt{|\Delta_F|}}\int_{R=1}^{+\infty}2\pi\frac{1}{R^{2s-1}}+\frac{O(1)}{R^{2s}}\cdot dR$$

$$=\frac{2}{w_F\sqrt{|\Delta_F|}}\frac{2\pi}{2-2s}\cdot R^{2-2s}\Big|_{R=1}^{+\infty}+\underbrace{\frac{1}{1-2s}R^{1-2s}\Big|_{R=1}^{+\infty}}$$

↑finite when $s\to 1$

$$=\frac{2}{w_F\sqrt{|\Delta_F|}}\cdot\frac{\pi}{s-1}+O(1)$$

For other ideal class group $[c]$, fix an ideal $I_c\in[c]$.

Every genuine ideal in $[c]$ takes the form $I_c\cdot(\alpha)$ for $\alpha\in I_c^{-1}\cdot O_F\setminus\{0\}$

$$\text{So} \sum_{0\neq I\subseteq[c]} \frac{1}{\|I\|^s} = \sum_{\alpha\in I_c^{-1}\mathcal{O}_F\setminus\{0\}} \frac{1}{\|I_c\|^s\cdot(N\alpha)^s} \overset{\substack{\text{same}\\ \text{argument}}}{=\!=} \underbrace{\frac{1}{\|I_c\|}}_{\frac{1}{\|I_c\|^s}\text{ as } s\to 1}\cdot\frac{2}{w_F\sqrt{|\Delta_F|}}\cdot\frac{\pi}{s-1}\cdot\underbrace{\|I_c\|}_{\substack{\downarrow\\ \text{density of}\\ \text{points increase by } \|I_c\|}} + O(1)$$

$$\underline{\text{Sum-up}}:\ \zeta_F(s) = \sum_{\substack{0\neq I\subseteq\mathcal{O}_F\\ \text{ideal}}} \frac{1}{\|I\|^s} = \frac{2\pi\cdot h_F}{w_F\sqrt{|\Delta_F|}}\cdot\frac{1}{s-1} + O(1)$$

② When F is real quadratic, similar to above, it (essentially) suffices to compute

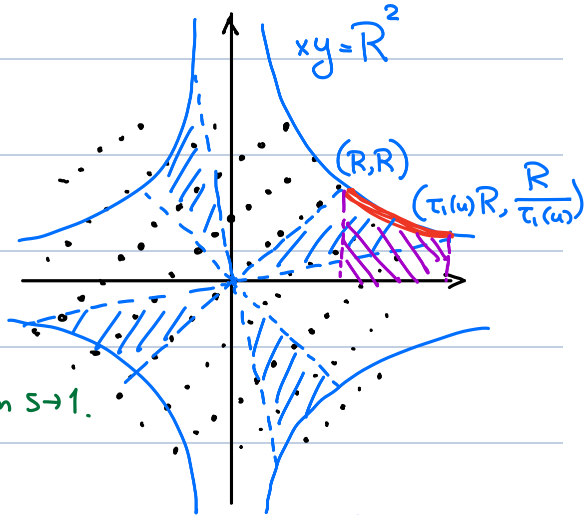
$$\sum_{\substack{0\neq I\subseteq\mathcal{O}_F\\ \text{principal}}} \frac{1}{\|I\|^s} = \sum_{a\in\mathcal{O}_F-\{0\}/\mathcal{O}_F^\times} \frac{1}{(Na)^s}$$

$$= \frac{1}{w_F}\int_{R=1}^{+\infty} \underbrace{\frac{1}{\sqrt{|\Delta_F|}}}_{\text{density}}\left(\frac{d}{dR}\left(\text{shaded area}(R)\right) + \underbrace{O(R)}_{\text{bounded when } s\to 1.}\right)\frac{1}{R^{2s}}\,dR$$

$$\text{shaded area} = 4\cdot\left(\frac{1}{2}R^2 - \frac{1}{2}\tau_1(u)R\cdot\frac{R}{\tau_1(u)} + \int_R^{\tau_1(u)R}\frac{R^2}{x}\,dx\right)$$

$$= 4\ R^2\cdot\ln|x|\Big|_R^{\tau_1(u)R} = 4R^2\cdot\left(\ln|\tau_1(u)R| - \ln|R|\right)$$

$$= 4\cdot R^2\ln|\tau_1(u)|$$



$\tau_1, \tau_2 : \mathcal{O}_F \hookrightarrow \mathbb{R}^2$

u = fundamental unit

assume $\tau_1(u) > 0$ for simplicity.

$$\Rightarrow \sum_{\substack{0\neq I\subseteq\mathcal{O}_F\\ \text{principal}}} \frac{1}{\|I\|^s} = O(1) + \frac{1}{w_F\sqrt{|\Delta_F|}}\cdot\int_{R=1}^{+\infty} 8R\cdot\ln|\tau_1(u)|\cdot\frac{1}{R^{2s}}\,dR$$

$$= O(1) + \frac{1}{w_F\sqrt{|\Delta_F|}}\cdot\frac{8\ln|\tau_1(u)|}{2-2s}\cdot R^{2-2s}\Big|_1^{+\infty}$$

$$= O(1) + \frac{1}{w_F\sqrt{|\Delta_F|}}\cdot\frac{\ln|\tau_1(u)|}{s-1} \quad ✓$$