Special values of L-functions 5

——— cyclotomic units

<u>Last time</u>. For F a number field with r_1 real embeddings and r_2 pairs of embeddings,

$$\lim_{s\to 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\frac{h_F \cdot \mathrm{Reg}_F}{w_F}, \quad \lim_{s\to 1}(s-1)\cdot\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \cdot \mathrm{Reg}_F}{w_F\,|\Delta_F|^{\frac{1}{2}}}$$

Today, we focus on the case when $F_N^+ = \mathbb{Q}(\zeta_N)^+ := \mathbb{Q}(\zeta_N+\zeta_N^{-1})$, the totally real subfield of $\mathbb{Q}(\zeta_N)$

$$\lim_{s\to 0} s^{-\frac{\varphi(N)}{2}+1} \zeta_{F_N^+}(s) = -\frac{h_{F_N^+}\cdot \mathrm{Reg}_{F_N^+}}{w_{F_N^+} \leftarrow 2}$$

$$\parallel$$

$$\zeta(0)\cdot \prod_{\substack{\eta \text{ even} \\ \eta\neq \mathbb{1}}} \lim_{s\to 0} s^{-1} L(\eta, s)$$

$$\Rightarrow \prod_{\substack{\eta \text{ even} \\ \eta\neq \mathbb{1}}} \lim_{s\to 0} s^{-1} L(\eta,s) = h_{F_N^+}\cdot \mathrm{Reg}_{F_N^+}$$

Similar story at $s=1$, $$\lim_{s\to 1}(s-1)\,\zeta_{F_N^+}(s) = \frac{2^{\varphi(N)/2}\cdot h_{F_N^+}\cdot \mathrm{Reg}_{F_N^+}}{2\cdot|\Delta_{F_N^+}|^{\frac{1}{2}}}$$

$$\parallel$$

$$\prod_{\substack{\eta \text{ even} \\ \eta\neq \mathbb{1}}} L(\eta, 1)$$

§1. Special values of even Dirichlet L-value at $s=1$.

Let η be a primitive Dirichlet character $\eta: (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$, $\eta(-1)=1$. <u>even</u>

$$L(\eta,1) = \sum_{n\geq 1} \frac{\eta(n)}{n}$$

Recall $\log(1-\zeta_M^i) = -\sum_{n\geq 1} \frac{\zeta_M^{in}}{n}$ ← function is modulo M.

Will make a Fourier transform:

$$\eta(n) = \frac{1}{M}\sum_{i=0}^{M-1} \zeta_M^{in}\Big(\underbrace{\sum_{j=0}^{M-1}\eta(j)\,\zeta_M^{-ij}}\Big)$$

$\hookrightarrow$ if $(i,M)=d\neq 1$, $\eta|_{(1+\frac{M}{d}\mathbb{Z}/M\mathbb{Z})^\times} \neq \mathbb{1} \Rightarrow$ sum $=0$

if $(i,M)=1$, this is $\eta(i)^{-1}G(\eta)$

So $$L(\eta,1) = \sum_{n\geq 1}\frac{\eta(n)}{n} = \frac{G(\eta)}{M}\sum_{n\geq 1}\frac{\zeta_M^{in}\cdot\eta(i)^{-1}}{n} = -\frac{G(\eta)}{M}\sum_{i=1}^{M-1}\eta(i)^{-1}\cdot\log(1-\zeta_M^i)$$

$$= -\frac{G(\eta)}{M}\sum_{i\in(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm1\}}\eta(i)^{-1}\log|1-\zeta_M^i|^2.$$

<u>Recall functional equation</u>: $$\underbrace{\Gamma\left(\tfrac{s}{2}\right)}_{\approx\frac{2}{s}}\cdot L(\eta^{-1},s) = G(\eta^{-1})\cdot\underbrace{M^{-s}}_{\approx 1}\cdot L(\eta,1-s)\cdot\underbrace{\pi^{s-\frac12}\,\Gamma\left(\tfrac{1-s}{2}\right)}_{\approx 1}$$

$$\Rightarrow \lim_{s\to0} s^{-1}\underbrace{L(\eta^{-1},s)}_{\substack{\| \\ L(\tilde\eta,s)}} = \frac{-G(\eta^{-1})G(\eta)}{2M}\cdot\sum_{i\in(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm1\}}\eta^{-1}(i)\log|1-\zeta_M^i|^2$$

as Galois rep'n

$$= -\sum_{i\in(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm1\}}\eta^{-1}(i)\log|1-\zeta_M^i|$$

In general, we are in the situation $M|N$, i.e. $\eta:(\mathbb{Z}/N\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \mathbb{Q}^{alg,\times}$

<u>Note</u>: say $N=L\cdot M \Rightarrow 1-\zeta_M^i = (1-\zeta_N^i)(1-\zeta_N^i\zeta_L)\cdots(1-\zeta_N^i\zeta_L^{L-1}) = \prod_{j=0}^{L-1}(1-\zeta_N^{i+jM})$

So, still we have $$\lim_{s\to0} s^{-1}L(\tilde\eta,s) = -\sum_{i\in(\mathbb{Z}/N\mathbb{Z})^\times/\{\pm1\}}\tilde\eta^{-1}(i)\log|1-\zeta_N^i|$$

<u>Remark</u>: $$\zeta_{\mathbb{Q}(\mu_N)^+}(s) = \zeta_{\mathbb{Q}}(s)\cdot\prod_{\substack{\tilde\eta:\mathrm{Gal}(\mathbb{Q}(\mu_N)^+/\mathbb{Q})\to\mathbb{C}^\times \\ \text{non-trivial}}}\underbrace{L(\tilde\eta,s)}$$

simple pole at $s=1$

$\llcorner$ so must be non-zero when $s=1$.

§2 Special values of p-adic even Dirichlet L-function at $s=1$

<u>p-adic analogue</u>! Assume $\eta : (\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \mathbb{Q}^{\mathrm{alg},\times} \subseteq \bar{\mathbb{Q}}_p^\times$ is primitive and $p \nmid M$, and even

Have constructed $\mu_\eta^{\{p\}} \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ s.t.

$$\int_{\mathbb{Z}_p^\times} \eta_p(x)\, x^n \, d\mu_\eta^{\{p\}}(x) = L^{\{p\}}(\eta\eta_p, -n)$$

When η is even, we compute: $\int_{\mathbb{Z}_p^\times} x^{-1} d\mu_\eta^{\{p\}}(x) =: L_p(\eta, 1)$

<u>Lemma</u>: For $\mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p)$, define $x\cdot\mu$ to be the measure

s.t. $\int_{\mathbb{Z}_p} f(x)\, d(x\mu)(x) = \int_{\mathbb{Z}_p} x f(x)\, d\mu(x)$.

Then $A_{x\cdot\mu}(T) = (1+T)\frac{d}{dT} A_\mu(T)$ In literature, often we write $\partial = (1+T)\frac{d}{dT}$

Proof: $A_{x\cdot\mu}(T) = \int_{\mathbb{Z}_p} x(1+T)^x d\mu = (1+T)\frac{d}{dT}\left(\int_{\mathbb{Z}_p} (1+T)^x d\mu\right) = (1+T)\frac{d}{dT}(A_\mu(T))$. $\square$

<u>Lemma</u>. If $\mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Z}_p)$ s.t. $\psi(\mu)=0$, i.e. $\mu \in \mathcal{D}_0(\mathbb{Z}_p^\times, \mathbb{Z}_p)$

define the measure $x^{-1}\mu$ by $\int_{\mathbb{Z}_p^\times} f(x)\, d(x^{-1}\mu)(x) = \int_{\mathbb{Z}_p^\times} x^{-1} f(x)\, d\mu(x)$.

Then. $A_{x^{-1}\mu}(T)$ is the unique power series $B \in \mathbb{Z}_p[\![T]\!]$

s.t. $(1+T)\frac{d}{dT}(B) = A_\mu(T)$, and $\psi(B)=0$

Proof: The first equality follows from the previous lemma.

Such B satisfies $\partial(B) = A_\mu(T)$ is unique up to adding a const.

The condition $\psi(B)=0$ determines that constant.

<u>Recall</u> $A_{\mu_\eta}(T) = \dfrac{\sum_{a=1}^{M-1} \eta(a)(1+T)^a}{1-(1+T)^M}$ and $\mu_\eta^{\{p\}} = \mathrm{Res}_{\mathbb{Z}_p^\times}(\mu_\eta)$

Apply "Fourier transform" to $A_{\mu_\eta}(T)$

$$A_{\mu_\eta}(T) = \sum_{a \geq 1} \eta(a)(1+T)^a = \frac{G(\eta)}{M} \sum_{i=1}^{M-1} \sum_{a \geq 1} \zeta_M^{ia} \cdot \eta(i)^{-1} (1+T)^a$$

$$= \frac{1}{G(\eta^{-1})} \sum_{i=1}^{M-1} \frac{\eta(i)^{-1}}{1 - \zeta_M^i (1+T)}$$

Then, we get $A_{\mu_\eta^{\{p\}}}(T) = \dfrac{1}{G(\eta^{-1})} \displaystyle\sum_{i=1}^{M-1} \left(\frac{\eta(i)^{-1}}{1 - \zeta_M^i (1+T)} - \frac{\eta(i)^{-1}}{1 - \zeta_M^{ip} (1+T)^p} \right)$

We take $B(T) \overset{\text{"}}{:=} - \dfrac{1}{G(\eta^{-1})} \displaystyle\sum_{i=1}^{M-1} \eta(i)^{-1} \left(\log_p\left(1 - \zeta_M^i (1+T)\right) - \frac{1}{p} \log\left(1 - \zeta_M^{ip} (1+T)^p\right) \right)$

$$= - \frac{1}{p\, G(\eta^{-1})} \sum_{i=1}^{M-1} \eta(i)^{-1} \log_p \left(\frac{\left(1 - \zeta_M^i (1+T)\right)^p}{1 - \zeta_M^{ip} (1+T)^p} \right)$$

$$\Vert$$

$$1 + \frac{\overbrace{\left(1 - \zeta_M^i (1+T)\right)^p - 1 + \zeta_M^{ip} (1+T)^p}^{\text{in } p\mathbb{Z}_p[[T]]}}{\underbrace{1 - \zeta_M^{ip} (1+T)^p}_{\text{unit in } \mathbb{Z}_p[[T]]}}$$

so $B(T)$ is well-defined in $\mathbb{Z}_p[[T]]$.

Next, we check $\psi(B(T)) = 0$, look at the $(1+T)$-adic expansion:

$$\log_p\left(1 - \zeta_M^i (1+T)\right) = - \sum_{a \geq 1} \frac{\zeta_M^{ia}}{a} (1+T)^a$$

$$\frac{1}{p} \log_p\left(1 - \zeta_M^{ip} (1+T)^p\right) = \sum_{b \geq 1} \frac{\zeta_M^{ipb}}{pb} (1+T)^{pb}$$

cancels the terms with $a = pb$.

$\Rightarrow B(T) = A_{x^{-1}\mu_\eta^{\{p\}}}(T)$

$$\Rightarrow L_p(\eta, 1) = \int_{\mathbb{Z}_p^\times} d\mu_B(x) = B(0) = - \frac{1}{p\, G(\eta^{-1})} \sum_{i=1}^{M-1} \eta(i)^{-1} \log_p \frac{(1 - \zeta_M^i)^p}{1 - \zeta_M^{ip}}$$

$$= - \frac{1}{G(\eta^{-1})} \cdot \left(1 - \eta(p) p^{-1}\right) \cdot \sum_{i=1}^{M-1} \eta(i)^{-1} \log_p\left(1 - \zeta_M^i\right). \qquad \square$$

Similar to the case of $L(\eta,1)$, we have for any nontrivial representation $\tilde{\eta}: Gal(\mathbb{Q}(\mu_N)^+/\mathbb{Q}) \to \overline{\mathbb{Q}}^{alg,\times}$, $p \nmid N$

$$L_p(\tilde{\eta},1) = -\frac{1}{G(\eta_{prim})}\left(1-\eta^{-1}(p)p^{-1}\right)\sum_{i=1}^{N-1}\eta(i)\log_p(1-\zeta_N^i)$$

↖ same works even if $p \mid N$

Remark: Why not consider $L_p(\tilde{\eta},0)$? No functional equation for $L_p(\eta)$??!!

$\mathcal{D}_0(\mathbb{Z}_p^\times,\mathbb{Z}_p) \simeq \mathbb{Z}_p[\Delta]\otimes\mathbb{Z}_p[\![(1+p\mathbb{Z}_p)^\times]\!] \cong \bigoplus_{\chi:\Delta\to\mathbb{Z}_p^\times}\mathbb{Z}_p[\![(1+p\mathbb{Z}_p)^\times]\!]$, $\Delta = \mathbb{F}_p^\times$.

If η is even, $L^{\{p\}}(\eta,\text{even})=0 \Rightarrow \mathcal{L}_p(\eta)_\chi = 0$ when χ is even

If η is odd, $L^{\{p\}}(\eta,\text{odd})=0 \Rightarrow \mathcal{L}_p(\eta)_\chi = 0$ when χ is odd.

§3. Cyclotomic units

Dirichlet unit theorem For F a number field with r_1 real embeddings and r_2 pairs of embeddings,

$$\mathcal{O}_F^\times = \underbrace{\mu(F)}_{\text{roots of unity in } F} \times \mathbb{Z}^{r_1+r_2-1}.$$

Definition For an abelian extension F of $\mathbb{Q}$, contained in $\mathbb{Q}(\mu_N)$,

Put $\mathcal{C}yc_F^\times := \mathcal{O}_F^\times \cap \langle -1, \zeta_N, \zeta_N^a - 1 ; a\in\mathbb{Z}\rangle$

It's called the subgroup of cyclotomic units.

Remark: This does not depend on the choice of N b/c $\forall M \mid N$, $Nm_{\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)}(\zeta_N^a - 1) = \zeta_M^a - 1$.

Example: If $(a,N)=1$, then $\dfrac{\zeta_N^a-1}{\zeta_N-1} \in \mathcal{O}_{F_N}^\times$ for $F_N = \mathbb{Q}(\mu_N)$.

Proof: $\dfrac{\zeta_N^a-1}{\zeta_N-1} = 1+\zeta_N+\dots+\zeta_N^{a-1} \in \mathcal{O}_{F_N}$

Find b s.t. $ab\equiv 1 \bmod N$.

$$\frac{\zeta_N-1}{\zeta_N^a-1} = \frac{\zeta_N^{ab}-1}{\zeta_N^a-1} = 1+\zeta_N^a+\dots+\zeta_N^{a(b-1)} \in \mathcal{O}_{F_N}. \quad \square$$

Theorem When $F^+ = \mathbb{Q}(\mu_{p^n})^+$, $\mathcal{C}yc_{F^+} = \left\langle \pm 1, \dfrac{\zeta_{p^n}^{a/2}-\zeta_{p^n}^{-a/2}}{\zeta_{p^n}^{1/2}-\zeta_{p^n}^{-1/2}}, 1<a<\dfrac{p^n}{2}, (a,p)=1 \right\rangle$

Remark: What about general F_N^+? $\qquad \zeta_{p^n}^{(a-1)/2} \cdot \frac{\zeta_{p^n}^a - 1}{\zeta_{p^n} - 1} =: \xi_a$

Proof: Note if $m<n$, $(a,p)=1$, $\zeta_{p^m}^a - 1 = \prod_{j=1}^{p^{n-m}} (\zeta_{p^n}^a \cdot \zeta_{p^{n-m}}^j - 1)$

$$(\zeta_{p^m}^{-a} - 1)/(\zeta_{p^m} - 1) = -\zeta_{p^m}^{a} \cdot (\zeta_{p^m}^{-a} - 1)/(\zeta_{p^m} - 1)$$

So $\mathcal{C}yc_{F^+}^\times = \left\langle -1, \zeta_{p^m}, \frac{\zeta_{p^m}^a - 1}{\zeta_{p^m} - 1} ; (a,p)=1, 1<a<\frac{p^m}{2} \right\rangle \cap \mathcal{O}_{F^+}^\times$

From this, we see $\mathcal{C}yc_{F^+}^\times = \langle -1, \xi_a ; 1<a<\frac{p^m}{2}, (a,p)=1 \rangle$. $\square$

Proposition $\left\{ \frac{\zeta_N^a - 1}{\zeta_N - 1} \,\middle|\, a \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}, a \notin \{\pm 1\} \right\}$ forms a $\mathbb{Q}$-basis of $\mathbb{Z}[\zeta_N]^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.

(Cor: $\mathcal{C}yc_{\mathbb{Q}(\mu_{p^m})^+}^\times \subseteq \mathbb{Z}[\zeta_{p^m}]^{+,\times}$ has finite index.)

Proof: Consider $\Psi : \left(\bigoplus_{a \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm1\}} \mathbb{Q} \cdot e_a \right)^{\mathrm{sum}=0} \longrightarrow \mathbb{Z}[\zeta_N]^\times \otimes_{\mathbb{Z}} \mathbb{Q}$

$$e_a \longmapsto \zeta_N^a - 1 \approx \zeta_N^{-a} - 1$$

$(\mathbb{Z}/N\mathbb{Z})^\times$-action $\quad \underset{\sim\sim\sim}{b \cdot e_a = e_{ab}} \quad \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$-action

Let $E = \mathbb{Q}(\text{images of all characters of } (\mathbb{Z}/N\mathbb{Z})^\times)$

$$\left(\bigoplus_{a \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm1\}} \mathbb{Z} \cdot e_a \right)^{\mathrm{sum}=0} \otimes E \cong \bigoplus_{\substack{\eta : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \\ \text{even, nontrivial}}} E \cdot \eta$$

basis element is $\sum_{b \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm1\}} \eta^{-1}(b)\, e_b =: e_\eta$?

It suffices to show that $e_\eta \neq 0$ in $E \cdot \eta$.

But note $0 \neq \lim_{s \to 0} s^{-1} L(\tilde{\eta}, s) = -\sum_{i \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm1\}} \eta^{-1}(i) \log |1 - \zeta_N^i|$ ✓ $\square$

Summary. Consider $M :=$ "$\mathrm{Spec}\, \mathbb{Q}(\mu_N)^+$" choose a p-adic place v of E

$$M_{\mathrm{et},p} = H^0_{\mathrm{\acute{e}t}}(\mathrm{Spec}\, \mathbb{Q}(\mu_N)^+ \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, E_v) \simeq E_v[\mathrm{Gal}(\mathbb{Q}(\mu_N)^+/\mathbb{Q})] \supseteq E_v \cdot \tilde{\eta}$$

$\rightsquigarrow L(\tilde{\eta}, s)$ and $\mathcal{L}_p(\tilde{\eta}, s)$

$\uparrow$ the p-adic measure interpolating $L^{\{p\}}(\tilde{\eta}\tilde{\eta}_p, 0)$

for $\tilde{\eta}_p : \mathrm{Gal}_{\mathbb{Q}}^{p\text{-ab}} \to \bar{\mathbb{Q}}_p^\times$ locally algebraic of nonpositive weight

Motivic element $\left(\mathcal{O}^\times_{\mathbb{Q}(\mu_N)^+} \otimes E\right)[\tilde{\eta}] \ni e_\eta$:

$$L(\tilde{\eta}, 1) = (*) \cdot \log e_\eta$$

$$\mathcal{L}_p(\tilde{\eta}\tilde{\eta}_p, 1) = (*) \cdot \log_p(e_{\eta\eta_p})$$

Key remark : Recall that p-adic L-function can be defined by interpolating values of $L^{\{p\}}(\tilde{\eta}\tilde{\eta}_p, -n)$ for a fixed n and all $\tilde{\eta}_p$.

So alternatively, $\mathcal{L}_p(\tilde{\eta})$ can be defined by interpolating values of $\log_p(e_\eta)$.