Special values of L-functions 6

—— (φ,Γ)-modules and Galois cohomology

Today, we make a digression on one version of p-adic Hodge theory: (φ,Γ)-modules

Goal: Just enough to understand Coleman's power series (next time)

§1. Galois representations for the Galois group of char p field.

- E field of char $p>0$, not necessarily perfect (in later example, $E=\mathbb{F}_p((t))$.)
- $\mathrm{Gal}_E := \mathrm{Gal}(E^{sep}/E)$
- $\varphi : E^{sep} \to E^{sep}$ $x \mapsto x^p$ the (arithmetic) Frobenius

Definition A φ-module over E is a finite dim'l E-vector space M together with an isomorphism $\Phi : M \otimes_{E,\varphi} E \xrightarrow{\sim} M$

(This is equivalent to give $\varphi : M \to M$ semilinear: i.e. $\varphi(am)=\varphi(a)\varphi(m)$ + matrix for φ being nondegenerate.)

Theorem. We have an equivalence of categories

$$\mathrm{Rep}_{\mathbb{F}_p}(\mathrm{Gal}_E) \xleftrightarrow{\sim} \varphi\text{-}\mathrm{Mod}/E$$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{F}_p} E^{sep})^{\mathrm{Gal}_E}$$

$$\mathbb{V}(D) := (D \otimes_E E^{sep})^{\varphi=1} \longleftarrow D$$

compatible with tensors and duals, + preserving dimensions

Proof: ① Let $d := \dim_{\mathbb{F}_p} V$.

Claim There is an E^{sep}-linear isomorphism $\underset{\circlearrowright\,\mathrm{Gal}_E}{V \otimes_{\mathbb{F}_p} E^{sep}} \simeq \underset{\circlearrowright\,\mathrm{Gal}_E}{(E^{sep})^{\oplus d}}$ equivariant for Gal_E-actions

"E^{sep} eats up all actions on V"

Proof: Pick a basis of V $\rightsquigarrow$ let $a_g \in GL_d(E) \subseteq GL_d(E^{sep})$ for the matrix of $g \in Gal_E$

Then $a_{gh} = a_g \cdot g(a_h)$

So this defines a 1-cocycle in $Z^1(Gal_E, GL_d(E^{sep}))$

By Hilbert 90, $H^1(Gal_E, GL_d(E^{sep})) = \{1\}$

$\Rightarrow \exists\, b \in GL_d(E^{sep})$ s.t. $a_g = b^{-1} g(b)$.

$\Rightarrow$ w.r.t. basis $(e_1, \dots, e_b) b^{-1}$, $\quad g(e \cdot b^{-1}) = e\, a_g\, g(b)^{-1} = e \cdot b^{-1}$ $\square$

So $\mathbb{D}(V) = (V \otimes_E E^{sep})^{Gal_E} \simeq E^d$; it carries an action from φ. Also $\mathbb{D}(V) \otimes_E E^{sep} \simeq V \otimes_{\mathbb{F}_p} E^{sep}$.

(2) Conversely, if $d = \dim D$ for $D \in \varphi\text{-Mod}/E$,

WTS $\dim_{\mathbb{F}_p} (D \otimes_E E^{sep})^{\varphi=1} = d$.

Pick a basis of D/E, and write $P \in GL_n(E)$ for matrix of φ.

<u>Want</u>: $\underline{v} \in (E^{sep})^n$ s.t. $P \cdot \varphi(\underline{v}) = \underline{v}$ or $\varphi(\underline{v}) - P^{-1}\underline{v} = 0$

$$\begin{pmatrix} v_1^p \\ \vdots \\ v_d^p \end{pmatrix} - P^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} = 0$$

$\Leftrightarrow$ E^{sep}-points of the algebra $\underbrace{E[v_1, \dots, v_n] / \varphi(\vec{v}) - P^{-1}(\vec{v})}$

This is a finite étale algebra by Jacobian criterion

So, has p^d-solutions $\Rightarrow D \otimes_E E^{sep} \simeq (E^{sep})^{\oplus d}$ as φ-modules

In particular, $\dim_{\mathbb{F}_p} (D \otimes_E E^{sep})^{\varphi=1} = d$. , and $D \otimes_E E^{sep} \simeq \mathbb{V}(D) \otimes_{\mathbb{F}_p} E^{sep}$

<u>Mutual inverses</u>: $\mathbb{D}(\mathbb{V}(D)) = (\mathbb{V}(D) \otimes_{\mathbb{F}_p} E^{sep})^{Gal_E} = (D \otimes_E E^{sep})^{Gal_E} = D$

$\mathbb{V}(\mathbb{D}(V)) = (\mathbb{D}(V) \otimes_E E^{sep})^{\varphi=1} = (V \otimes_{\mathbb{F}_p} E^{sep})^{\varphi=1} = V$. $\square$

<u>Definition</u> A <u>Cohen ring</u> C_E of a field E of char p is a complete DVR with residue field k s.t. p is a uniformizer. (In later example, $C_E = A_{\mathbb{Q}_p} := \mathbb{Z}_p((T))^{\wedge, p\text{-adic}}$)

Remark A Cohen ring exists but may not be unique if E is not perfect.

Notation. Let C_E be a Cohen ring which admits a lift φ of the Frobenius on E

(In later example, $\varphi(T) = (1+T)^p - 1$.)

Then $\exists$ max unram ext'n C_E^{ur} of C_E s.t. $C_E^{ur}/(p) = E^{sep}$ (b/c $(C_E, (p))$ is Henselian)

Theorem. There is an equivalence of tensor categories

$$\underline{\mathrm{Rep}}_{\mathbb{Z}_p}(\mathrm{Gal}_E) \xleftrightarrow{\sim} \varphi\text{-}\mathrm{Mod}^{et}/C_E \quad \leftarrow \text{need not be free}/C_E$$

need not be free $/\mathbb{Z}_p$

$$V \longmapsto \mathbb{D}(V) = (V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}})^{\mathrm{Gal}_E}$$

$$\mathbb{V}(D) = (D \otimes_C \widehat{C^{ur}})^{\varphi=1} \longleftarrow D$$

Here *étale* means $\Phi: D \otimes_{C_E, \varphi} C_E \xrightarrow{\sim} D$, or equivalently, $\varphi: D \to D$ is semilinear, and the matrix for φ belongs to $GL_d(C_E)$. "integrally invertible"

Interpretation of Galois cohomology

For $V \in \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(\mathrm{Gal}_E)$, we can compute its Galois cohomology $H^1(\mathrm{Gal}_E, V)$ using (φ, Γ)-modules

$$0 \to \mathbb{F}_p \to E^{sep} \xrightarrow{\varphi - 1} E^{sep} \to 0 \quad \text{(Artin-Schreier exact sequence)}$$

$$\Rightarrow \quad 0 \to \mathbb{Z}_p \to \widehat{C_E^{ur}} \xrightarrow{\varphi - 1} \widehat{C_E^{ur}} \to 0$$

$$\xRightarrow{V \otimes_{\mathbb{Z}_p} -} \quad 0 \to V \to V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \xrightarrow{\varphi -} V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \to 0$$

Note: $V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \cong \mathbb{D}(V) \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}} \leftarrow$ is a "regular representation of $\mathrm{Gal}(E^{sep}/E)$"

Taking Galois cohomology $\Rightarrow$

$$0 \to H^0(\mathrm{Gal}_E, V) \to \underbrace{(V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}})^{\mathrm{Gal}_E}}_{= \mathbb{D}(V)} \xrightarrow{\varphi - 1} \underbrace{(V \otimes_{\mathbb{Z}_p} \widehat{C_E^{ur}})^{\mathrm{Gal}_E}}_{= \mathbb{D}(V)} \to H^1(\mathrm{Gal}_E, V) \to 0$$

$$\mathbb{D}(V) \xrightarrow{\varphi - 1} \mathbb{D}(V)$$

Fact: $R\Gamma(\mathrm{Gal}_E, V) \cong [\mathbb{D}(V) \xrightarrow{\varphi - 1} \mathbb{D}(V)]$

§2 (φ,Γ)-modules vs. representations of $\mathrm{Gal}_{\mathbb{Q}_p}$.

Big Key Fact

$$\begin{tikzcd}
\overline{\mathbb{Q}}_p \arrow[d, dash] & & \mathbb{F}_p((T))^{sep} \arrow[d, dash] \\
\mathbb{Q}_p(\mu_{p^\infty}) \arrow[d, dash] & H_{\mathbb{Q}_p} \cong \mathrm{Gal}_{\mathbb{F}_p((t))} & \mathbb{F}_p((T)) =: E \\
\mathbb{Q}_p & \Gamma \cong \mathbb{Z}_p^\times \ni a,\ \gamma_a \leftarrow\!\shortmid a & \Gamma\text{-action } \gamma_a(T) = (1+T)^a - 1
\end{tikzcd}$$

Choose $C_E = A_{\mathbb{Q}_p} := \mathbb{Z}_p((T))^{\wedge, p\text{-adic}}$, φ-action on C_E: $\varphi(T) = (1+T)^p - 1$.

Theorem We have an equivalence of tensor categories

$$\mathrm{Rep}^{(\text{free})}_{\mathbb{F}_p / \mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p}) \xleftrightarrow{\ \sim\ } (\varphi,\Gamma)\text{-Mod}^{\text{ét},\ (\text{free})} / E = \mathbb{F}_p((T)) \ /\ C_E = A_{\mathbb{Q}_p}$$

$$V \longmapsto \mathbb{D}(V) := \left(V \otimes_{\mathbb{Z}_p} \widehat{A^{ur}_{\mathbb{Q}_p}}\right)^{H_{\mathbb{Q}_p}}$$

$$\left(D \otimes_{A_{\mathbb{Q}_p}} \widehat{A^{ur}_{\mathbb{Q}_p}}\right)^{\varphi=1} \longleftarrow\!\shortmid D$$

Remark: Theory works for p-adic repn's of Gal_K for K CDVF of mixed char + perfect residue field.

Will later explain a special case for $\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}$

Definition An étale (φ,Γ)-module is a finitely generated $A_{\mathbb{Q}_p}$-mod M, equipped with continuous semilinear Γ-action ($\gamma(am) = \gamma(a)\gamma(m)$), and a Γ-equivariant isomorphism

$$\Phi : M \otimes_{A_{\mathbb{Q}_p}, \varphi} A_{\mathbb{Q}_p} \xrightarrow{\sim} M$$

(étale $\Leftrightarrow$ matrix for φ-action is integrally invertible.)

Example: Let $\mathcal{O}$ = ring of integers in a finite extension of $\mathbb{Q}_p$.

If $\chi : \mathrm{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}^{ab}_{\mathbb{Q}_p} \simeq \widehat{\mathbb{Q}_p^\times} = p^{\hat{\mathbb{Z}}} \times \mathbb{Z}_p^\times \longrightarrow \mathcal{O}^\times$ continuous character of $\mathrm{Gal}_{\mathbb{Q}_p}$,

($p \mapsto$ geometric Frobenius ϕ)

want to compute $\mathbb{D}(\chi)$.

① $\chi(\phi) = 1$. $\quad \mathbb{D}(\chi) = \left(\mathcal{O}\cdot e_\chi \otimes_{\mathbb{Z}_p} \widehat{A^{ur}_{\mathbb{Q}_p}}\right)^{H_{\mathbb{Q}_p}} = A_{\mathbb{Q}_p} \cdot e_\chi$

$$\varphi(e_\chi) = e_\chi \ , \quad \gamma_a(e_\chi) = \chi(a)\, e_\chi$$

② $\chi(p)$ general

$$\begin{tikzcd}
 & \overline{\mathbb{Q}}_p \arrow[d, dash] \arrow[dl, dash, bend right, "H_{\mathbb{Q}_p}"'] & & \mathbb{F}_p((t))^{sep} \arrow[d, dash] \\
 & \mathbb{Q}_p^{ur}(\mu_{p^\infty}) \arrow[d, dash] & H^{inert}_{\mathbb{Q}_p} \simeq I_E & \overline{\mathbb{F}}_p((t)) \\
\mathbb{Q}_p(\mu_{p^\infty}) \arrow[ur, dash] \arrow[d, dash] & & E = \mathbb{F}_p((t)) \arrow[ur, dash] & \\
\mathbb{Q}_p \arrow[r, dash] & \mathbb{Q}_p^{ur} & &
\end{tikzcd}$$

$$\mathbb{D}(\chi) = \left(\mathcal{O}\cdot e_\chi \otimes_{\mathbb{Z}_p} \widehat{\mathbb{A}^{ur}_{\mathbb{Q}_p}}\right)^{H_{\mathbb{Q}_p}} = \left(\mathcal{O}\cdot e_\chi \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{ur}((T))^{\wedge, p\text{-adic}}\right)^{Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)}$$

Hilbert 90' for $\overline{\mathbb{F}}_p/\mathbb{F}_p$ or $\widehat{\mathbb{Z}_p^{ur}}/\mathbb{Z}_p \implies$

$\exists\, \lambda \in \widehat{\mathbb{Z}_p^{ur}}$ s.t. for geom. Frobenius $\phi \in Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$,

$$\phi(\lambda) = \chi(p)^{-1}\cdot\lambda$$

Then $\phi(e_\chi \otimes \lambda) = \chi(p)\, e_\chi \otimes \chi(p)^{-1}\cdot \lambda = e_\chi \otimes \lambda$

So $\mathbb{D}(\chi) = \mathbb{A}_{\mathbb{Q}_p,\mathcal{O}}\cdot(e_\chi\otimes\lambda)$

$$\varphi(e_\chi\otimes\lambda) = e_\chi \otimes \phi^{-1}(\lambda) = \chi(p)\cdot e_\chi\otimes\lambda \ , \quad \gamma_a(e_\chi\otimes\lambda) = \chi(a)\, e_\chi\otimes\lambda.$$

↑ ϕ^{-1} is arithmetic Frobenius

<u>Conclusion</u>: The (φ,Γ)-module attached to the Galois character

$\chi : Gal_{\mathbb{Q}_p} \twoheadrightarrow Gal^{ab}_{\mathbb{Q}_p} \simeq \widehat{\mathbb{Q}_p^\times} \longrightarrow \mathcal{O}^\times$ is

$$\mathbb{A}_{\mathbb{Q}_p}(\chi) := \mathbb{A}_{\mathbb{Q}_p,\mathcal{O}}\cdot e\ , \quad \varphi(e) = \chi(p)\, e\ , \quad \gamma_a(e) = \chi(a)\, e$$

<u>Remark</u>: Although φ looks like an arithmetic Frobenius, it's information $\longleftrightarrow$ geometric Frob.

§3 <u>Galois cohomology in terms of (φ,Γ)-modules</u>.

Keep the notation as above. Let $V \in \underline{Rep}_{\mathbb{Z}_p}(Gal_{\mathbb{Q}_p})$.

$$\begin{tikzcd}
\overline{\mathbb{Q}}_p \arrow[d, dash, "H_{\mathbb{Q}_p}"] \\
\mathbb{Q}_p(\mu_{p^\infty})
\end{tikzcd}$$

<u>Serre spectral sequence</u>: $H^i(\Gamma, H^j(H_{\mathbb{Q}_p}, V)) \Rightarrow H^{i+j}(Gal_{\mathbb{Q}_p}, V)$

<u>A better way</u>: let $\gamma\in\Gamma$ be a topological generator

Γ
$\mathbb{Q}_p$

$$R\Gamma(H_{\mathbb{Q}_p}, V) \cong [\mathbb{D}(V) \xrightarrow{\varphi-1} \mathbb{D}(V)]$$

$$R\Gamma(\mathrm{Gal}_{\mathbb{Q}_p}, V) \simeq R\Gamma(\Gamma, R\Gamma(H_{\mathbb{Q}_p}, V))$$
$$= R\Gamma\left(\Gamma, [\mathbb{D}(V) \xrightarrow{\varphi-1} \mathbb{D}(V)]\right)$$
$$= \left[\begin{tikzcd} \mathbb{D}(V) \arrow[r, "\varphi-1"] \arrow[d, "\gamma-1"] & \mathbb{D}(V) \arrow[d, "\gamma-1"] \\ \mathbb{D}(V) \arrow[r, "\varphi-1"] & \mathbb{D}(V) \end{tikzcd}\right]$$

$$C_{\varphi,\gamma} : \mathbb{D}(V) \xrightarrow{(\varphi-1,\gamma-1)} \mathbb{D}(V) \oplus \mathbb{D}(V) \xrightarrow{(\gamma-1, 1-\varphi)} \mathbb{D}(V)$$ Herr's complex.

$$x \longmapsto ((\varphi-1)x, (\gamma-1)x)$$

$$(x,y) \longmapsto (\gamma-1)x + (1-\varphi)y$$

Its cohomology $H^i_{\varphi,\Gamma}(\mathbb{D}(V)) \cong H^i(\mathrm{Gal}_{\mathbb{Q}_p}, V)$.

<u>A nontrivial feature</u>: ψ-operator!

<u>Recall</u>: $\varphi : \mathbb{A}_{\mathbb{Q}_p} = \mathbb{Z}_p((T))^{\wedge, p\text{-adic}} \hookrightarrow \mathbb{A}_{\mathbb{Q}_p}$ $\qquad \varphi(T) = (1+T)^p - 1$

There is a canonical decomposition as $\varphi(\mathbb{A}_{\mathbb{Q}_p})$-modules

$$\mathbb{A}_{\mathbb{Q}_p} = \bigoplus_{i=0}^{p-1} (1+T)^i \varphi(\mathbb{A}_{\mathbb{Q}_p}) \qquad (*)$$

Now, for a (φ,Γ)-module D over $\mathbb{A}_{\mathbb{Q}_p}$, we may tensor $(*)$ with D over $\varphi(\mathbb{A}_{\mathbb{Q}_p})$ to get

$$\begin{tikzcd} D \otimes_{\mathbb{A}_{\mathbb{Q}_p}, \varphi} \mathbb{A}_{\mathbb{Q}_p} \arrow[r, phantom, "="] \arrow[d, phantom, "\wr\|\ \Phi"] & \bigoplus_{i=0}^{p-1} (1+T)^i \varphi(\mathbb{A}_{\mathbb{Q}_p}) \otimes_{\varphi(\mathbb{A}_{\mathbb{Q}_p})} \mathbb{A}_{\mathbb{Q}_p} \arrow[d, phantom, "\wr\|"] \\ D \arrow[r, phantom, "\simeq"] & \bigoplus_{i=0}^{p-1} (1+T)^i \varphi(D) \end{tikzcd}$$

So, every element $x \in D$ can be written uniquely as $x = \sum_{i=0}^{p-1} (1+T)^i \varphi(x_i)$

Define $\psi(x) := x_0$; it is Γ-equivariant.

Theorem. $H^i(\mathrm{Gal}_{\mathbb{Q}_p}, V)$ can be computed using (ψ,Γ)-cohomology

$$C_{\psi,\gamma}(V) : \mathbb{D}(V) \xrightarrow{(\psi-1,\gamma-1)} \mathbb{D}(V) \oplus \mathbb{D}(V) \xrightarrow{(\gamma-1,\,1-\psi)} \mathbb{D}(V).$$

Proof:

$$\begin{tikzcd}
 & & \mathbb{D}(V)^{\psi=0} \arrow[r, "\gamma-1"] \arrow[d] & \mathbb{D}^{\psi=0} \arrow[d] \\
C_{\varphi,\gamma}(V): & \mathbb{D}(V) \arrow[r, "(\varphi-1,\gamma-1)"] \arrow[d, equal] & \mathbb{D}(V)\oplus\mathbb{D}(V) \arrow[r, "(\gamma-1,\,1-\varphi)"] \arrow[d, "-\psi\ \ \|"'] & \mathbb{D}(V) \arrow[d, "-\psi"'] \\
C_{\psi,\gamma}(V): & \mathbb{D}(V) \arrow[r, "(\psi-1,\gamma-1)"] & \mathbb{D}(V)\oplus\mathbb{D}(V) \arrow[r, "(\gamma-1,\,1-\psi)"] & \mathbb{D}(V)
\end{tikzcd}$$

① $-\psi(\varphi-1) = -1+\psi$

② $\gamma-1$ acts invertibly on $\mathbb{D}(V)^{\psi=0}$. $\square$

Upshot: ψ-operator (not linear!) behaves much better for cohomology theory.