Special values of L-functions 7

—— Coleman power series

Recall: Have introduced p-adic L-functions, and proved that (today, will only consider trivial tame part)

$$\begin{tikzcd} & L(\eta_p,1) \\ \exists \text{ "linear combination" of } p\text{-power cyclotomic units} \arrow[ur, "\text{usual log}"] \arrow[dr, "p\text{-adic } \log_p"'] & \\ & \zeta_p(\eta_p,1) \end{tikzcd}$$

for $\eta_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{Q}^{alg,\times}$

Key remark: p-adic ζ-function can be determined by values of $L(\eta_p,-n)$ for a fixed n, but varying η_p

Expectation p-adic ζ-function can be reconstructed from cyclotomic units.

Today's picture:

$$\begin{tikzcd} \boxed{\text{system of motivic elements}} \arrow[r] & \boxed{\text{classes in the Iwasawa cohomology}} \arrow[r] \arrow[dr, "\substack{\text{next time}\\ \text{Euler system argument}}"'] & \boxed{p\text{-adic L-functions}} \arrow[d, leftrightarrow, "\text{Iwasawa Main Conjecture}"] \\ & & \boxed{\text{upper bounds on the "dual" Iwasawa cohomology}} \end{tikzcd}$$

§1. Kummer theory

Let F be a field of char $\neq p$

$$1 \to \mu_{p^n} \to (F^{sep})^\times \xrightarrow{x \mapsto x^n} (F^{sep})^\times \to 1$$

Take Gal_F-cohomology $\Rightarrow$ $F^\times/(F^\times)^{p^n} \simeq H^1(Gal_F, \mu_{p^n})$

If E/F is a finite extension,

$$\begin{tikzcd} E^\times/(E^\times)^{p^n} \arrow[r, "\sim"] \arrow[d, "\text{norm}"] & H^1(Gal_E,\mu_{p^n}) \arrow[d, "\text{cores}"] \\ F^\times/(F^\times)^{p^n} \arrow[r, "\sim"] & H^1(Gal_F,\mu_{p^n}) \end{tikzcd} \qquad (*)$$

Example: F_v finite over $\mathbb{Q}_p$, take inverse limit

$$\Rightarrow F^\times \hat{\otimes} \mathbb{Z}_p \simeq H^1(Gal_F, \mathbb{Z}_p(1))$$

Variant: F global field. $S \in M_F$ finite set of places

Write $F^{\langle S\rangle}$ for the max'l extension unramified outside S. Put $\mathrm{Gal}_{F,S} := \mathrm{Gal}(F^{\langle S\rangle}/F)$

When $S = \{\text{all } p\text{-adic and } \infty\text{-places of } F\}$, write $F^{\langle p\infty\rangle}$ and $\mathrm{Gal}_{F,p\infty}$ instead

We have $1 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{O}_{F^{\langle p\infty\rangle}}[\frac{1}{p}]^\times \xrightarrow{x \mapsto x^{p^n}} \mathcal{O}_{F^{\langle p\infty\rangle}}[\frac{1}{p}]^\times \longrightarrow 1$

This surjective is not very trivial.

$$\Rightarrow \mathcal{O}_F[\tfrac{1}{p}]^\times / p^n \hookrightarrow H^1(\mathrm{Gal}_{F,p\infty}, \mu_{p^n})$$

alternative notation: $H^1(\mathcal{O}_F[\frac{1}{S}], \mu_{p^n})$

Taking inverse limit $\Rightarrow \mathcal{O}_F[\frac{1}{p}]^\times \hat{\otimes} \mathbb{Z}_p \hookrightarrow H^1(\mathrm{Gal}_{F,p\infty}, \mathbb{Z}_p(1))$ as in $H^1_{et}(\mathrm{Spec}\, \mathcal{O}_F[\frac{1}{S}], \mu_{p^n})$

Remark: ① We need $\mathrm{Gal}_{F,p}$ for finiteness; can do the same for $\mathrm{Gal}_{F,S}$.

② Without Hilbert 90', we only have injectivity.

§2. Iwasawa cohomology.

or Gal_F if F is local.

Consider the Galois extension $F(\mu_{p^\infty})/F$ with $\Gamma_F := \mathrm{Gal}(F(\mu_{p^\infty})/F)$, and $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{F,S})$.

$$\begin{array}{c} F(\mu_{p^\infty}) \\ | \\ F(\mu_{p^r}) \\ | \\ F \end{array} \qquad \Gamma_F,\ \Gamma_{F,r}$$

We define the Iwasawa cohomology to be

$$H^1_{Iw}(\mathrm{Gal}_{F,S}, V) := \varprojlim_{r,\ \mathrm{cores}} H^1(\mathrm{Gal}_{F(\mu_{p^r}),S}, V)$$

inverse limit under corestriction map

Shapiro's lemma

$$\cong \varprojlim_r H^1(\mathrm{Gal}_{F,S}, V \otimes \mathbb{Z}_p[\mathrm{Gal}(F(\mu_{p^r})/F)])$$

$$\cong H^1(\mathrm{Gal}_{F,S}, V \otimes \mathbb{Z}_p[\![\Gamma_F]\!])$$

This Iwasawa cohomology is a module over $\mathbb{Z}_p[\![\Gamma_F]\!]$.

Two possible points of view: ① Take inverse limit as above.

② View it as the cohomology of cyclotomically deformed representation.

Main example today: Take the inverse limit of the following diagram.

$$
\begin{tikzcd}
\mathbb{Z}[\zeta_{p^r}][\tfrac{1}{p}]^\times \hat{\otimes} \mathbb{Z}_p \arrow[r, "\text{Kummer}"] \arrow[d] & H^1(\mathrm{Gal}_{\mathbb{Q}(\mu_{p^r}),p\infty}, \mathbb{Z}_p(1)) \arrow[d] \\
\mathbb{Q}_p(\zeta_{p^r})^\times \hat{\otimes} \mathbb{Z}_p \arrow[r, "\text{Kummer}", "\simeq"'] & H^1(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}, \mathbb{Z}_p(1))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\varprojlim_{\text{norm}} \mathbb{Z}[\zeta_{p^r}][\tfrac{1}{p}]^\times \hat{\otimes} \mathbb{Z}_p \arrow[r] \arrow[d] & H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q},p\infty}, \mathbb{Z}_p(1)) \arrow[d] \\
\varprojlim_{\text{norm}} \mathbb{Q}_p(\zeta_{p^r})^\times \hat{\otimes} \mathbb{Z}_p \arrow[r, "\simeq"] & H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{Z}_p(1))
\end{tikzcd}
$$

inverse system of cyclotomic units : $(\zeta_{p^r} - 1)_{r \in \mathbb{Z}_{\geq 1}}$

§3. Iwasawa cohomology in terms of ψ-operators

Recall : There is an equivalence $\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p}) \simeq (\varphi,\Gamma)\text{-}\mathrm{Mod}^{\text{ét}}/A_{\mathbb{Q}_p}$ coming from

$$
\begin{tikzcd}
 & \overline{\mathbb{Q}}_p \arrow[d, dash, "H_{\mathbb{Q}_p} \cong \mathrm{Gal}_E"] & \mathbb{F}_p((T))^{\text{sep}} \arrow[d, dash] \\
(1+p^r\mathbb{Z}_p)^\times = \Gamma_r & \mathbb{Q}_p(\mu_{p^\infty}) \arrow[dl, dash] \arrow[dd, dash, bend left, "\Gamma = \mathbb{Z}_p^\times"] & \mathbb{F}_p((T)) = E \quad \Gamma \ni \gamma_a \\
\boxed{\mathbb{Q}_p(\mu_{p^r})} \arrow[dr, dash] & & \\
 & \mathbb{Q}_p &
\end{tikzcd}
$$

For any $r \in \mathbb{Z}_{\geq 1}$, we have similarly, $\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}) \simeq (\varphi,\Gamma_r)\text{-}\mathrm{Mod}^{\text{ét}}/A_{\mathbb{Q}_p}$

We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p}) \arrow[r, "\mathbb{D}(-)"] \arrow[d, bend right, "\mathrm{Res}"'] & (\varphi,\Gamma)\text{-}\mathrm{Mod}^{\text{ét}}/A_{\mathbb{Q}_p} \arrow[d, bend right, "\mathrm{Res}"'] \\
\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}) \arrow[r, "\mathbb{D}(-)"] \arrow[u, bend right, "\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}}^{\mathrm{Gal}_{\mathbb{Q}_p}}(-)"'] & (\varphi,\Gamma_r)\text{-}\mathrm{Mod}^{\text{ét}}/A_{\mathbb{Q}_p} \arrow[u, bend right, "\mathrm{Ind}_{\Gamma_r}^{\Gamma}(-)"']
\end{tikzcd}
$$

Recall the isomorphism of cohomology $H^i(\mathrm{Gal}_{\mathbb{Q}_p}, V) \cong H^i_{\varphi,\gamma}(\mathbb{D}(V)) \cong H^i_{\psi,\gamma}(\mathbb{D}(V))$

where $H^i_{*,\gamma}(\mathbb{D}(V))$ is defined by $\mathbb{D}(V) \xrightarrow{(*-1,\gamma-1)} \mathbb{D}(V)\oplus\mathbb{D}(V) \xrightarrow{(\gamma-1,1-*)} \mathbb{D}(V)$

A version for $V_r \in \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}(\mu_{p^r})})$, γ_{1+p^r} is the topological generator of Γ_r

then $H^i(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}, V_r)$ is isomorphic to $H^i_{\varphi,\gamma_{1+p^r}}(\mathbb{D}(V_r)) \cong H^i_{\psi,\gamma_{1+p^r}}(\mathbb{D}(V_r))$

For $V \in \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(\mathrm{Gal}_{\mathbb{Q}_p})$, the restriction and corestriction map can be computed by

$$\begin{tikzcd}
H^i(\mathrm{Gal}_{\mathbb{Q}_p}, V) \arrow[d, shift left, "\mathrm{res}"] & C_{\psi,\gamma}: & \mathbb{D}(V) \arrow[r, "(\psi-1,\gamma-1)"] \arrow[d, shift left, "\mathrm{id}"] & \mathbb{D}(V)\oplus\mathbb{D}(V) \arrow[r, "(\gamma-1,1-\psi)"] \arrow[d, shift left, "{\mathrm{id},\ \sum_{i=0}^{(p-1)p^{r-1}-1}\gamma^i}"] & \mathbb{D}(V) \arrow[d, shift left, "\sum_{i=0}^{(p-1)p^{r-1}-1}\gamma^i"] \\
H^i(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}, V) \arrow[u, shift left, "\mathrm{cores}"] & C_{\psi,\gamma_{1+p^r}}: & \mathbb{D}(V) \arrow[r, "(\psi-1,\gamma_{1+p^r}-1)"] \arrow[u, shift left, "\mathrm{id}"] & \mathbb{D}(V)\oplus\mathbb{D}(V) \arrow[r, "(\gamma_{1+p^r}-1,1-\psi)"] \arrow[u, shift left, "{\sum_{i=0}^{(p-1)p^{r-1}-1}\gamma^i,\ \mathrm{id}}"] & \mathbb{D}(V) \arrow[u, shift left, "\mathrm{id}"]
\end{tikzcd}$$

<u>Lemma</u>. The cohomology group $H^1(\mathrm{Gal}_{\mathbb{Q}_p(\mu_{p^r})}, V) \cong H^1_{\psi,\gamma_{1+p^r}}(\mathbb{D}(V))$ sits in an exact sequence.

$$0 \to \mathbb{D}(V)^{\psi=1}/(\gamma_{1+p^r}-1) \to H^1_{\psi,\gamma_{1+p^r}}(\mathbb{D}(V)) \to \left(\mathbb{D}(V)/\psi-1\right)^{\gamma_{1+p^r}=1} \to 0 \quad (*)$$

Proof: The map is given by
$$B \longmapsto (0,B)$$
$$(A,B) \longmapsto A \qquad \square$$

<u>Notation</u> $\Gamma = \mathbb{Z}_p^\times \simeq \Delta \times (1+p\mathbb{Z}_p)^\times$ with $\Delta = \mathbb{F}_p^\times$

$\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[\Delta]\otimes\mathbb{Z}_p[[X]]$, $\quad X = [\exp(p)]-1$.

Put $\mathbb{Z}_p((\Gamma)) \cong \mathbb{Z}_p[\Delta]\otimes\mathbb{Z}_p((X))^{\wedge, p\text{-adic}}$

<u>Big Theorem</u>. Let $D \in (\varphi,\Gamma)\text{-}\mathrm{Mod}^{\text{ét}}/A_{\mathbb{Q}_p}$. $\mathrm{rank}_{A_{\mathbb{Q}_p}} D =: r$. We have

$$\begin{tikzcd}
\Gamma \curvearrowright & D^{\psi=1} \arrow[r, "1-\varphi"] & D^{\psi=0} \\
& \mathbb{Z}_p[[\Gamma]] \arrow[u, phantom, "\circlearrowleft"] \arrow[r] & \mathbb{Z}_p((\Gamma)) \arrow[u, phantom, "\circlearrowleft"]
\end{tikzcd}$$

(<u>note</u>: $\psi(x)=x \Rightarrow \psi(\varphi(x)-x) = x-\psi(x)=0$)

① $D^{\psi=1}$ is a finitely generated $\mathbb{Z}_p[[\Gamma]]$-mod, s.t. $\mathrm{rank}_{\mathbb{Z}_p[[\Gamma]]} D^{\psi=1} = r$, ② $D^{\psi=0}$ is a free $\mathbb{Z}_p((\Gamma))$-module of rank r.

③ $D/\psi-1$ is a finitely generated $\mathbb{Z}_p$-module (not just $\mathbb{Z}_p[[\Gamma]]$)

<u>Corollary</u> The Iwasawa cohomology $\mathrm{Exp}: H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q}_p}, V) \cong \mathbb{D}(V)^{\psi=1}$.

Proof: Consider the inverse limit defining Iwasawa cohomology

fin. gen. $\mathbb{Z}_p$-mod.

$$\begin{tikzcd}
0 \arrow[r] & \boxed{\mathbb{D}(V)^{\psi=1}} / (\gamma_{1+p^{r+1}}-1) \arrow[r] \arrow[d, "\text{natural}"] & H^1_{\psi,\gamma_{1+p^{r+1}}}(\mathbb{D}(V)) \arrow[r] \arrow[d, "\text{corestriction}"] & \left(\boxed{\mathbb{D}(V)/\psi-1}\right)^{\gamma_{1+p^{r+1}}=1} \arrow[r] \arrow[d, "\sum_{i=0}^{p-1}\gamma^i_{1+p^{r+1}}"] & 0 \\
0 \arrow[r] & \mathbb{D}(V)^{\psi=1} / (\gamma_{1+p^{r}}-1) \arrow[r] & H^1_{\psi,\gamma_{1+p^{r}}}(\mathbb{D}(V)) \arrow[r] & \left(\mathbb{D}(V)/\psi-1\right)^{\gamma_{1+p^{r}}=1} &
\end{tikzcd}$$

fin. gen. $\mathbb{Z}_p[[\Gamma]]$-module

← approximately p when $r \to \infty$

In the limit $\mathrm{Exp}: H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q}_p}, V) \cong \mathbb{D}(V)^{\psi=1}$. □

§4 <u>Coleman power series</u>

<u>Main Theorem</u> We have the following.

$$(\zeta_{p^r}-1)_{r\geq 1} \mapsto -\frac{1+T}{T}$$

$$\begin{tikzcd}
\varprojlim_{\text{norm}} \mathbb{Z}[\zeta_{p^r}]^\times \subseteq & \varprojlim_{\text{norm}} \mathbb{Z}[\zeta_{p^r}][\frac{1}{p}]^\times \arrow[r] \arrow[d, "\text{Kummer}"] & \varprojlim_{\text{norm}} \mathbb{Q}_p(\zeta_{p^r})^\times \arrow[d, "\text{Kummer}", "\wr"'] & & \\
& H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q},p\infty}, \mathbb{Z}_p(1)) \arrow[r, "\mathrm{loc}_p"] & H^1_{Iw}(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{Z}_p(1)) \arrow[r, "\mathrm{Exp}", "\cong"'] & \mathbb{Z}_p((T))^{\wedge,\psi=1} \arrow[r, "1-\varphi"] & \mathbb{Z}_p((T))^{\wedge,\psi=0} \\
& & & \mathbb{Z}_p[[T]]^{\psi=1} \arrow[u, phantom, "\cup"] \arrow[r, "1-\varphi"] & \mathbb{Z}_p[[T]]^{\psi=0} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$\left(\frac{\zeta^a_{p^r}-1}{\zeta_{p^r}-1}\right)_{r\geq 1} \in \varprojlim_{\text{norm}} \mathbb{Z}[\zeta_{p^r}]^\times$$

from $\mathbb{Z}_p(1)$

$$\mapsto (1-a\gamma_a)\left(-\frac{1+T}{T}\right) = -\frac{1+T}{T} + \frac{a(1+T)^a}{(1+T)^a-1} \mapsto \zeta_{p,a}$$

p-adic zeta

<u>This follows from the following construction of Coleman's power series</u>

Recall $\mathbb{Z}_p[[T]]$ is a finite free $\varphi(\mathbb{Z}_p[[T]])$-module of degree p,

$$\exists \text{ norm map}: \mathcal{N}: \mathbb{Z}_p[\![T]\!]^\times \longrightarrow \varphi(\mathbb{Z}_p[\![T]\!])^\times \xleftarrow[\varphi]{\sim} \mathbb{Z}_p[\![T]\!]^\times$$

$$f(T) \longmapsto \prod_{i=0}^{p-1} f((1+T)\zeta_p^i - 1)$$

$\left(\mathcal{N}(T) = T\right.$ b/c $T \mapsto \prod_{i=0}^{p-1}((1+T)\zeta_p^i - 1) = (1+T)^p - 1 = \varphi(T).)$

Then we have the following isomorphisms of exact sequences:

$$\begin{tikzcd}
\left(\frac{\zeta_{p^r}^a - 1}{\zeta_{p^r} - 1}\right)_{r\geq 1} & 0 \arrow[r] & \varprojlim_{\text{norm}} \mathbb{Z}_p[\zeta_{p^r}]^\times \arrow[r] & \varprojlim_{\text{norm}} \mathbb{Q}_p(\zeta_{p^r})^\times \arrow[r, "-v_{\zeta_{p^r}-1}(-)"] & \mathbb{Z} \arrow[r] & 0 \\
\frac{(1+T)^a - 1}{T} \arrow[u, mapsto] \arrow[d, mapsto] & 1 \arrow[r] & \mathbb{Z}_p[\![T]\!]^{\times, \mathcal{N}=1} \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\cong"] & \mathbb{Z}_p[\![T]\!][\frac{1}{T}]^{\times, \mathcal{N}=1} \arrow[r, "\mathrm{val}_T"] \arrow[u, "\mathrm{Col}\ \cong"', "f(T) \mapsto (f(\zeta_{p^r}-1))_{r\geq 1}"] \arrow[d, "f(T) \mapsto (1+T)\frac{d}{dT}\log f(T)"] & \mathbb{Z} \arrow[r] \arrow[u, equal] \arrow[d] & 0 \\
-\frac{1+T}{T} + \frac{a(1+T)^a}{(1+T)^a - 1} & 0 \arrow[r] & \mathbb{Z}_p[\![T]\!]^{\psi=1} \arrow[r] & \mathbb{Z}_p(\!(T)\!)^{\psi=1} \arrow[r, "\mathrm{Res}_{T=0}"] & \mathbb{Z}_p \arrow[r] & 0
\end{tikzcd}$$

Remark: We did not verify that this is the natural isomorphism from (φ, Γ)-modules.

Proof: ① For Col: $f(T) \longmapsto (f(\zeta_{p^r} - 1))_{r\geq 1}$,

$$\mathcal{N}f = f \Rightarrow f((1+T)^p - 1) = \prod_{i=0}^{p-1} f(\zeta_p(1+T) - 1)$$

Plug in $T = \zeta_{p^{r+1}} - 1 \Rightarrow f(\zeta_{p^r} - 1) = \prod_{i=0}^{p-1} f(\zeta_p \zeta_{p^r} - 1) = \mathrm{Nm}_{\mathbb{Q}_p(\mu_{p^{r+1}})/\mathbb{Q}_p(\mu_{p^r})} f(\zeta_{p^r} - 1)$.

It is injective by Weierstrass preparation theorem

② Inverse of Col: we need some lemmas

(1) If $\varphi(f)(T) \equiv 1 \bmod p^k$ for some $k \geq 0$, then $f(T) \equiv 1 \bmod p^k$

($\Leftrightarrow \varphi(f) \equiv 0 \bmod p^k \Rightarrow f \equiv 0 \bmod p^k$, b/c φ is injective mod p)

(2) For $f \in \mathbb{Z}_p[\![T]\!]^\times$, we have $\mathcal{N}(f) \equiv f \bmod p$

(b/c when mod p, $\varphi = \mathrm{Frob}$, the norm map is $f \mapsto f^p$. So $\mathcal{N}(f) \equiv f \bmod p$.)

(3) If $f \equiv 1 \bmod p^k$, then $\mathcal{N}(f) \equiv 1 \bmod p^{k+1}$

Write $f(T) = 1 + p^k \cdot g(T)$, $\varphi(\mathcal{N}(f))(T) = \prod_{j=0}^{p-1}(1 + p^k g(\zeta_p^j(1+T) - 1))$

$$= 1 + p^k \cdot p\, \psi(g)(\varphi(T)) + \text{higher terms}.$$

Done by (1).

(4) If $f \in \mathbb{Z}_p[\![T]\!]^\times$ and $k_2 \geq k_1 \geq 0$, then $N^{k_2}(f) \equiv N^{k_1}(f) \mod p^{k_1+1}$

From $\dfrac{N^{k_2-k_1} f}{f} \equiv 1 \mod p$, iterate (3). ✓.

Existence of f_u s.t. $\mathrm{Col}(f_u) = u_n$ for $(u_n) \in \varprojlim_{\text{norm}} \mathbb{Z}_p[\zeta_{p^n}]^\times$

For each n, choose $f_n \in \mathbb{Z}_p[\![T]\!]^\times$ s.t. $f_n(\zeta_{p^n} - 1) = u_n$

Put $g_n = N^{2n} f_n$.

Then for $m \geq n$, $g_m(\zeta_{p^n} - 1) = (N^{2m} f_m)(\varphi^{m-n}(\zeta_{p^m} - 1))$

$$\equiv (N^{m-n} f_m)(\varphi^{m-n}(\zeta_{p^m} - 1)) \mod p^{m-n}$$

$$= Nm_{\mathbb{Q}(\mu_{p^m})/\mathbb{Q}(\mu_{p^n})} f_m(\zeta_{p^m} - 1) = Nm(u_m) = u_n.$$

So $\lim_{m \to +\infty} g_m(\zeta_{p^n} - 1) = u_n$. Find a convergent subsequence of g_m. ✓. $\square$

③ Given $f \in \mathbb{Z}_p[\![T]\!]^{\times, N=1}$, $\quad f((1+T)^p - 1) = \prod_{j=0}^{p-1} f(\zeta^j (1+T) - 1)$

$$\log f((1+T)^p - 1) = \sum_{j=0}^{p-1} \log f(\zeta^j (1+T) - 1)$$

$\frac{d}{dT} \downarrow$

$$(d\log f)((1+T)^p - 1) \cdot p(1+T)^{p-1} = \sum_{j=0}^{p-1} (d\log f)(\zeta^j (1+T) - 1) \cdot \zeta^j$$

$$= (1+T)^p \cdot (d\log f)((1+T)^p - 1) = \frac{1}{p} \sum_{j=0}^{p-1} \zeta^j (1+T) (d\log f)(\zeta^j (1+T) - 1)$$

So $(1+T)\, d\log f \in \mathbb{Z}_p[\![T]\!]^{\psi=1}$