Fall 2024: Topics in Number Theory: Special Values of L-functions

Instructor: Liang Xiao (肖梁)
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Meeting time: Tuesday 5–6 (odd weeks) and Thursday 7–8
Lecture room: Lecture Building #1, Room 204 (一教 204)
Office hours: by appointment
Webpage: http://bicmr.pku.edu.cn/~lxiao/24fall/24fall.htm

Goal of this course: The course will focus on the arithmetic properties of special values of L-functions. More precisely, we hope to cover the following aspects in this course:

(1) Kubota–Leopoldt p-adic L-function.
(2) Cyclotomic units and Euler system argument for Iwasawa Main Conjecture.
(3) General definition of L-functions, and Tate thesis
(4) Deligne's conjecture on special values of L-functions and periods, and periods of modular forms.
(5) A quick introduction to Waldspurger formula and Gross–Zagier formula.
(6) Introduction to Beilinson Conjecture, and examples.

I do not know of a good reference that covers all material. I will be hopefully latexing notes while posting hand-written notes.

Prerequisite:

- Very well versed with algebraic number theory, such as adeles and ideles, knows the statement of class field theory, and comfortable for talking about Galois cohomology.
- Expect to be quite familiar with Tate's thesis.
- Solid algebraic geometry (Hartshorne Chapter 2 and 3); some exposure to étale cohomology.
- Modular forms; better to have some exposure to adelic version, but not completely necessary.
- Reasonably familiar with elliptic curves, and some exposure to the concept of abelian varieties.

Grade Distribution:
Homeworks: 60%, due on Thursdays of Week 3, 6, 8, 10, 12, 14, 16, in total 7 times, with lowest grade dropped.

Take-home final exam: 40%, to be announced (probably one or two French-style long problems).

Homework: *Homework problems will be posted on the course webpage.* You are welcome and encouraged to work with other students on the problems, but you should write up your homework independently.

Syllabus (Tentative)

Lecture	Dates	Content
1	9/10	Introduction and special values of Dirichlet L-functions (algebraicity)
2	9/12	Kummer congruences and p-adic analysis over $\mathbb{Z}_p$
3	9/19	p-adic Dirichlet L-functions, and L-functions for Galois representations
4	9/24	Analytic class number formula
5	9/26	Cyclotomic units, regulators
Happy National's Day!		
6	10/8	(φ, Γ)-modules and Galois cohomology.
7	10/10	Coleman power series.
8	10/17	Iwasawa Main Conjecture.
9	10/22	Iwasawa Main Conjecture II.
10	10/24	A light introduction to motives.
11	10/31	Hodge structures and Deligne's conjecture
12	11/5	Periods of Hecke characters, and p-adic analogue
13	11/7	L-functions attached to modular forms, periods.
14	11/14	Introduction to Waldspurger's formula? (Xiao away, sub by Yuan)
15	11/19	Weak Mordell–Weil theorem for elliptic curves.
16	11/21	Height of points on an elliptic curve.
17	11/28	Introduction to Gross–Zagier's formula (LX away, sub by Yuan)
18	12/3	Overflow
19	12/5	Introduction to Beilinson's conjecture.
20	12/12	Examples of Beilinson's conjecture: K_2 of modular curves

21	12/17	Examples of Beilinson's conjecture: Rankin–Selberg case
22	12/19	Borel's regulator theorem
23	12/26	An overview of development on the study of special values of L-functions
	TBA	**Final Exam**