Topics in Number theory: Introduction to p-adic Hodge theory
Exercise 1 (due on October 9)

Choose 5 out of 7 problems to submit (The problems are chronically ordered by the materials, not necessarily by difficulties. I do recommend to at least read all problems.)

Exercise 1.1 ("Canonical forms" of Weil–Deligne representations)**.** Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let (r, N, V) be a Weil–Deligne representation.

(1) (Optional) There is a well-defined Frobenius semisimplification (r^{ss}, N, V) such that for any $g \in \mathrm{Gal}_K$, we have $r^{\mathrm{ss}}(g) = r(g)^{\mathrm{ss}}$. (Note: I only have an ugly proof of this.)

(2) Let $|\cdot|_K : \mathrm{Gal}_K \to \mathbb{C}^\times$ be the the character $|g| = q^{\mathrm{val}}(g)$.

There are two building blocks for Frobenius semisimple Weil–Deligne representations: (a) a Frobenius semisimple representation $\rho : \mathrm{W}_K \to \mathrm{GL}(U)$ with trivial monodromy operator $N = 0$; (b) $\mathrm{sp}(n) = \mathbb{C}^{\oplus n} = |\cdot|_K^{(1-n)/2} \oplus |\cdot|_K^{(3-n)/2} \oplus \cdots \oplus |\cdot|_K^{(n-1)/2}$ and

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then any Frobenius Weil–Deligne representation can be written (noncanonically) as a direct sum

$$(r, N) = \bigoplus_{i=1}^{r} (r_i, 0) \otimes \mathrm{sp}(n_i),$$

where $r_i : \mathrm{W}_K \to \mathrm{GL}(U_i)$ is a continuous Frobenius semisimple representation of W_K and $n_i \in \mathbb{Z}_{\geq 1}$.

(It might be possible to get a general description of Weil–Deligne representations without the Frobenius semisimple condition.)

Exercise 1.2 (WD$(-)$ is well defined)**.** Let ℓ and p be two distinct prime numbers. Fix an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and let Gal_K, W_K, I_K, and $\mathrm{I}_K^{\mathrm{t}}$ denote the Galois group, Weil group, inertia subgroup, and the tame inertia quotient of K. We choose a geometric Frobenius element ϕ_K and a nontrivial homomorphism $t_\ell : \mathrm{I}_K^{\mathrm{t}} \to \mathbb{Q}_\ell$. We choose an element τ of I_K such that $t_\ell(\tau) = 1$.

Let $\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Define

$$N := \frac{1}{m t_\ell(\tau)} \log(\rho(\tau)^m) \in \mathrm{End}(V)(-1), \tag{1.2.1}$$

for m sufficiently divisible (which then is independent of the choice of τ as long as $t_\ell(\tau) = 1$).

Define the associated Weil–Deligne representation associated to ρ to be (r, N, V): for $x \in \mathrm{I}_K$,

$$r(\phi_K^a \cdot x) := \iota_\ell\big(\rho(\phi_K^a \cdot x) \exp(-t_\ell(x) \cdot N)\big).$$

(Here $V = \mathbb{C}^{\oplus n}$ is the space for the representation r.)

(1) Check that (r, N) is a Weil–Deligne representation, i.e. r is a representation of W_K, and for $g \in \mathrm{Gal}_K$, $r(g) N r(g)^{-1} = q^{-\mathrm{val}(g)} N$, where $\mathrm{val} : \mathrm{W}_K \to \phi_K^{\mathbb{Z}} \cong \mathbb{Z}$ is the natural map.

(2) Suppose that we choose another geometric Frobenius element ϕ'_K and $t' : \mathrm{I}^{\mathrm{t}}_K \to \mathbb{Q}_\ell$ another nontrivial homomorphism. Then the new $\mathrm{WD}'(\rho)$ is isomorphic to $\mathrm{WD}(\rho)$. (But $\mathrm{WD}(\rho)$ depends on the isomorphism ι_ℓ.) (This will use part of Exercise 1.1.)

Exercise 1.3 (Weierstrass preparation theorem)**.** Let $(R, \mathfrak{m})$ be a complete local ring. Consider a power series in one variable x over R: $f(x) = \sum\limits_{n \geq 0} c_n x^n$ with $c_n \in R$. Suppose that there exists some $s \geq 0$ such that $c_s \in R \backslash \mathfrak{m}$.

There exists a unique factorization:

$$f(x) = U(x) \cdot P(x),$$

where $U(x) \in R[\![x]\!]^\times$ is a unit, and $P(x) \in R[x]$ is a monic polynomial such that $P(x) \equiv x^N$ modulo $\mathfrak{m}[x]$ for some N.

Exercise 1.4 (Uniqueness of extension of norms)**.** Let K be a complete valued field. In class, we have defined a valuation on K^{alg}: if $\alpha \in K^{\mathrm{alg}}$ is contained in a finite extension L of K, then

$$v(\alpha) = \frac{1}{[L : K]} v(N_{L/K}(\alpha)).$$

Show that any valuation on K^{alg} extending $v(-)$ on K is equal to the one above.

Exercise 1.5 (Newton above Hodge)**.** Let K be a complete valued field. Let $A = (a_{ij})$ be an $n \times n$-matrix with values in K.

(1) The Hodge polygon of A is the convex hull of points

$$\big(m,\, \min\{v(\text{all } m \times m\text{-minors})\}\big), \qquad m \in \{1, \dots, n\},$$

denote by $\mathrm{HP}(A)$. Show that if $P, Q \in \mathrm{GL}_n(\mathcal{O}_K)$, then $\mathrm{HP}(PAQ) = \mathrm{HP}(A)$.

(2) The Newton polygon $\mathrm{NP}(A)$ of A is defined to be the Newton polygon of the characteristic power series $\det(I_n - At)$. Show that the Newton polygon of A lies above the Hodge polygon of A.

(3) Suppose that A has Hodge polygon with slopes $r_1 \leq r_2 \leq \cdots \leq r_n$ (counted with multiplicity). By standard argument, show that A can be written as PDQ with $P, Q \in \mathrm{GL}_n(\mathcal{O}_K)$ and $D = \mathrm{diag}\{d_1, \dots, d_n\}$ a diagonal matrix such that $v(d_i) = r_i$.

(4) Conversely, suppose that $r_1 \leq r_2 \leq \cdots \leq r_n$ are real numbers, and suppose that we have

$$v(a_{ij}) \geq r_i, \qquad \text{for } i, j \in \{1, \dots, n\}.$$

Then $\mathrm{HP}(A)$ lies above the polygon with slopes $r_1, r_2, \dots, r_n$. (In particular, $\mathrm{NP}(A)$ lies above this polygon with slopes $r_1, r_2, \dots, r_n$.

Exercise 1.6 (Duality between Newton polygon and valuation polygon)**.** Let $f(t) = \sum\limits_{n \in \mathbb{Z}} a_n t^n$ is a formal Laurent series with values in K. Suppose that $\mathrm{NP}(f)$ has vertices whose x-coordinates tend to both $-\infty$ and ∞.

Recall that for $r \in \mathbb{R}$, we define

$$v_r(f) := \inf_{n \in \mathbb{Z}} \big(v(a_n) + nr\big)$$

whenever it is defined (e.g. if $f \in \mathcal{H}(I)$, $v_r(f)$ is defined for $r \in I$.

(1) Prove that $r \mapsto v_r(f)$ is a concave polygon.

(2) Show that we may recover $\mathrm{NP}(f)$ from the polygon $r \mapsto v_r(f)$.

Exercise 1.7 (Bounded function with infinitely many zeros)**.** Let $\mathbb{C}_p := \widehat{\mathbb{Q}_p^{\mathrm{alg}}}$. Construct a function $f = \sum_{n\geq 0} a_n t^n \in \mathcal{B}((0,+\infty])$, i.e. a bounded analytic function on the open unit disc that has infinitely many zeros on the open unit disc.