Introduction to p-adic Hodge theory 1

—— Introduction and motivation, Weil-Deligne representations

§1. Motivation from complex Hodge theory

- X projective smooth variety over a field k of dim m. (assume char $k=0$)

  $\rightsquigarrow \Omega^1_{X/k}$ = sheaf of differentials, $\Omega^i_{X/k} = \Lambda^i \Omega^i_{X/k}$

  de Rham complex $\Omega^\bullet_{X/k} = [\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots \to \Omega^m_{X/k}]$

  $\rightsquigarrow$ de Rham cohomology $H^n_{dR}(X/k) = \mathbb{H}^n(X, \Omega^\bullet_{X/k})$

For $i \geq 0$, there's a natural decreasing filtration on $H^n_{dR}(X/k)$:

$$\begin{tikzcd}
F^i\Omega^\bullet_{X/k} = \arrow[d] & {[} 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] & \Omega^i_{X/k} \arrow[r, "d"] \arrow[d, "id"] & \Omega^{i+1}_{X/k} \arrow[r, "d"] \arrow[d, "id"] & \cdots \arrow[r] & \Omega^m_{X/k} {]} \arrow[d] \\
\Omega^\bullet_{X/k} = & {[} \mathcal{O}_X \arrow[r] & \cdots \arrow[r] & \Omega^{i-1}_{X/k} \arrow[r] & \Omega^i_{X/k} \arrow[r] & \Omega^{i+1}_{X/k} \arrow[r] & \cdots \arrow[r] & \Omega^m_{X/k} {]}
\end{tikzcd}$$

Define $F^iH^n_{dR}(X/k) := \mathrm{Im}(\mathbb{H}^n(X, F^i\Omega^\bullet_{X/k}) \to \mathbb{H}^n(X, \Omega^\bullet_{X/k}))$

Fact: $\mathrm{gr}^i_F H^n_{dR}(X/k) \cong H^{n-i}(X, \Omega^i_{X/k})$ $\leftarrow$ a consequence of degeneration of de Rham spectral sequence.

Rmk: Why not

$$\begin{tikzcd}
{[} \mathcal{O}_X \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^i_X \arrow[r] \arrow[d, "id"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 {]} \\
{[} \mathcal{O}_X \arrow[r] & \cdots \arrow[r] & \Omega^i_X \arrow[r] & \Omega^{i+1}_X \arrow[r] & \cdots & {]}
\end{tikzcd}$$

This does not commute.

Now assume $k \subseteq \mathbb{R} \subseteq \mathbb{C}$

Then we have a comparison $H^n_B(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\sim} H^n_{dR}(X/k) \otimes_k \mathbb{C}$

complex conjugation actions: F_∞ on $H^n_B(X(\mathbb{C})^{an}, \mathbb{Z})$, σ on $\mathbb{C}$, σ on $\mathbb{C}$

$$F_\infty \otimes \sigma \longleftrightarrow 1 \otimes \sigma$$

Rmk: It is necessary to tensor with $\mathbb{C}$: $\mathbb{C}$ is called a "period ring"

$$H_n^B(X(\mathbb{C})^{an}, \mathbb{Z}) \times H_{dR}^n(X/k) \longrightarrow \mathbb{C}$$
$$(\gamma \quad , \quad \omega) \longmapsto \int_\gamma \omega$$

- Now, if K is a complete DVF of mixed characteristic $(0,p)$ with perfect residue field k.

  X projective smooth variety$/K$

Grothendieck's mysterious functor conjecture

$\exists$ a natural p-adic period ring B_{dR} such that

it is equiped a Gal_K-action + a natural decreasing filtration

$$\underset{\substack{\circlearrowleft \\ Gal_K \\ \uparrow \\ \text{no filtration}}}{H_{et}^n(X_{\bar{K}}, \mathbb{Z}_p)} \underset{\mathbb{Z}_p}{\otimes} \underset{\substack{\circlearrowleft \\ Gal_K \\ Fil}}{B_{dR}} \simeq \underset{\substack{\uparrow \\ \text{no } Gal_K\text{-action} \\ \text{Hodge Fil}}}{H_{dR}^n(X/K)} \underset{K}{\otimes} \underset{\substack{\circlearrowleft \\ Gal_K \\ Fil}}{B_{dR}}$$

Note: LHS has $\mathbb{Q}_p$-str. RHS has K-str. very different!

is an isomorphism of B_{dR}-v.s. with compatible Gal_K-action & filtrations.

Attempts: say $K = \mathbb{Q}_p$

① Period ring $B = \mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$. does not work.

e.g. $X = \mathbb{P}^1$, $H_{et}^2(\mathbb{P}^1_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) = \mathbb{Q}_p(-1) \circlearrowleft Gal_{\mathbb{Q}_p}$ acts by χ_{cycl}^{-1}

($\zeta_{p^m} = p^m$th primitive root of unity, $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$

$\mu_{p^\infty} := \{\text{all } p^{th} \text{ power roots of unity}\} = \langle \zeta_{p^m} ; m \in \mathbb{N} \rangle$

$\chi_{cycl} : Gal_K \twoheadrightarrow Gal(K(\mu_{p^\infty})/K) \longrightarrow \mathbb{Z}_p^\times$

s.t. $g(\zeta_{p^m}) = \zeta_{p^m}^{\chi_{cycl}(g)}$.

For a p-adic rep'n V, write $V(n) := V \otimes \chi_{cycl}^{\otimes n}$.

$\mathbb{Q}_p(-1)$ is a $\mathbb{Q}_p$-v.s. where Gal_K acts by χ_{cycl}^{-1}.)

Theorem (Tate–Sen) $\mathbb{C}_p(-1) \not\cong \mathbb{C}_p$ as semilinear rep'n of Gal_K.

related to $H^1(\mathrm{Gal}_K, \overline{K}^\times) = \{1\}$ but $H^1(\mathrm{Gal}_K, \hat{\overline{K}}^\times) \neq \{1\}$.

② Hodge–Tate period ring $\mathbb{B}_{HT} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$. (almost correct)

works for $\mathbb{Q}_p(m)$, but <u>not</u> for $H^1_{et}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$.

③ correct period ring (due to Fontaine) $\mathbb{B}_{dR}$

- $\mathbb{B}_{dR}$ is a complete DVF with uniformizer $t \in \mathbb{B}_{dR}$, & residue field $\mathbb{C}_p$

  $\Rightarrow$ abstractly, $\mathbb{B}_{dR} \approx \mathbb{C}_p((t))$ but <u>NOT</u> Gal_K-equivariantly.

- $\mathbb{B}_{dR}^+ :=$ valuation ring, $\mathrm{Fil}^i \mathbb{B}_{dR} := t^i \mathbb{B}_{dR}^+$.

§2. <u>Motivation from Galois representations</u>.

* <u>Philosophy</u>: ① To understand a group, e.g. $\mathrm{Gal}_{\mathbb{Q}_p}$, it is "equivalent" to understand its rep'ns (and their tensors)

② ℓ-adic rep'ns ($\ell \neq p$) are easier (later today)

③ Want to have an easier way to describe p-adic representations.

View comparison isom. $\underbrace{H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)}_{\text{a Galois rep'n}} \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR} = H^n_{dR}(X/K) \otimes_K \mathbb{B}_{dR}$

$\leadsto$ In general, for a p-adic Galois rep'n V of Gal_K, define $D_{dR}(V) := (V \otimes \mathbb{B}_{dR})^{\mathrm{Gal}_K}$

This "extracts" information out from p-adic rep'ns.

<u>Related question</u>: $X/\mathbb{Q}_p$ proj sm. var. can we compare $H^n_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ for ℓ a prime

- If X extends to proj. sm. var $\mathfrak{X}/\mathbb{Z}_p$

  When $\ell \neq p$, $H^n_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell) \simeq H^n_{et}(\mathfrak{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is unramified $\circlearrowleft \mathrm{Frob}_p$.

  $\mathrm{Tr}(\mathrm{Frob}_p^r)$ are determined by $\#\mathfrak{X}(\mathbb{F}_{p^r})$.

  $\Rightarrow H^n_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ are isomorphic to each other (after replacing Frob_p by its semisimplification)

General conjecture: $H^n_{et}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ is indep of ℓ. What does this mean? (explain today)

Question: What about $\ell = p$? (rest of this semester)

§3. Structure of Galois group of a CDVF

- K = complete discrete valuation field $\supseteq \mathcal{O}_K$ = valuation ring $\twoheadrightarrow k$ = residue field

  $\varpi_K \in \mathcal{O}_K$, ϖ_K = uniformizer. say char $k = p$.

$$\begin{array}{ll} K^{sep} & \\ \text{wild} \mid &) \; P_K = \text{wild inertia group, pro-}p\text{ group.} \\ K^{tame} = K^{ur}(\sqrt[n]{\varpi_K} ; p \nmid n) & \\ \text{tame} \mid &) \; \mathrm{Gal}(K^{tame}/K^{ur}) =: I_K^{tame} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \cong \varprojlim_{n, \text{mult}} \mu_n \\ K^{ur} = K(\zeta_n ; p \nmid n) \text{ if } k \text{ is a finite field} & \\ \text{unram.} \mid &) \; \mathrm{Gal}(K^{ur}/K) \simeq \mathrm{Gal}(k^{sep}/k) \simeq \hat{\mathbb{Z}} \\ K & \qquad \mathrm{Fr}_{\#k} \longleftarrow 1 \end{array}$$

$\prod_{\ell \neq p} \mathbb{Z}_\ell(1) = \hat{\mathbb{Z}}^{(p)}(1)$

$$\Psi : I_K^{tame} \xrightarrow{\sim} \varprojlim_{n, \text{mult}} \mu_n$$
$$g \longmapsto \frac{g(\varpi_K^{1/n})}{\varpi_K^{1/n}}$$

This does not depend on the choice of $\varpi_K^{1/n}$.

<u>Cor</u>: When k is finite ($\Leftarrow K$ is a finite ext'n of $\mathbb{Q}_p$), Gal_K is (pro-)solvable

(i.e. an inverse limit of finite solvable groups.)

<u>Remark</u>: We write $t_\ell : I_K^{tame} \cong \hat{\mathbb{Z}}^{(p)}(1)$ because $\mathrm{Gal}(K^{ur}/K)$ acts on it.

$\forall \sigma \in \mathrm{Gal}(K^{ur}/K)$, lift it to $\tilde{\sigma} \in \mathrm{Gal}(K^{tame}/K)$

then for $g \in I_K^{tame}$, define $\sigma(g) := \tilde{\sigma} g \tilde{\sigma}^{-1}$

This corresponds to the action of $\tilde{\sigma}$ on roots of unity b/c

$$\tilde{\sigma} g \tilde{\sigma}^{-1} \longmapsto \frac{\tilde{\sigma} g \tilde{\sigma}^{-1}(\varpi^{1/N})}{\varpi^{1/N}} = \tilde{\sigma}\left(\frac{g\tilde{\sigma}^{-1}(\varpi^{1/N})}{\tilde{\sigma}^{-1}(\varpi^{1/N})}\right) = \tilde{\sigma}\left(\frac{g\varpi^{1/N}}{\varpi^{1/N}}\right) = \tilde{\sigma}(\Psi(g))$$

Ψ independent of $\varpi^{1/N}$

Rmk: If char $k = 0$, $P_K = \{1\}$ and $K^{tame} = K^{ur}(\sqrt[n]{\varpi_K}, n \geq 1)$ and $I_K^{tame} = \hat{\mathbb{Z}}(1)$.

§4 Weil-Deligne representations

Assume that K is a finite extension of $\mathbb{Q}_p$. $K \supseteq \mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/(\varpi_K) = k$, $q := \# k$

<u>Definition</u> The <u>Weil group</u> W_K is defined by the pullback

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_K & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \mathrm{Gal}(\overline{k}/k) = \mathrm{Fr}_q^{\hat{\mathbb{Z}}} & \longrightarrow & 1 \\ & & \| & & \cup\!\!| & & \cup\!\!| & & \\ 1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathrm{Fr}_q^{\mathbb{Z}} & \longrightarrow & 1 \end{array}$$

Here Fr_q = geometric Frobenius = inverse of arithmetic Frobenius (raising to q^{th} power), $\mathrm{val}: W_K \longrightarrow \mathrm{Fr}_q^{\mathbb{Z}} \simeq \mathbb{Z}$, $\mathrm{Fr}_q \longmapsto 1$.

Fix $\ell \neq p$ and an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$

- Let $\rho: W_K \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ be a continuous ℓ-adic representation.

(<u>Remark</u>: What's the difference between representations of Galois groups and Weil groups?
If Fr_K is a Frobenius element, for representation of Galois groups,
we need $\rho(\mathrm{Fr}_K)$ to be topologically quasi-unipotent.
i.e. $\exists M \in \mathbb{Z}_{>0}$, s.t. $\rho(\mathrm{Fr}_K^M) \in I_n + \ell \cdot M_n(\overline{\mathbb{Z}}_\ell)$.)

Back to the independence of ℓ:

<u>naive idea</u>: $W_K \xrightarrow{\rho} GL_n(\overline{\mathbb{Q}}_\ell) \overset{\iota_\ell}{\simeq} GL_n(\mathbb{C})$ but this is not continuous.

Issue: ℓ-part of the tame inertia $\longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ $\rightsquigarrow$ N linear operator

- Preview of p-adic Hodge theory for Galois rep'ns:

* <u>ℓ-adic story</u>:

$$\begin{array}{c} \overline{K} \\ | \\ K^{tame} \\ | \\ K^{ur} \\ | \\ K \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right) \ \ell\text{-adic theory relies on this}$$

* p-adic story

$$\begin{tikzcd}[arrows=dash]
\overline{K} \arrow[d] \arrow[ddr, bend left, "{\mathrm{Gal}(\overline{K}/K(\mu_{p^\infty})) \simeq \mathrm{Gal}_{k_K((\pi_K))}\left(k_K((\pi_K))^{sep} \,\big|\, k_K((\pi_K))\right)}"] & \\
K^{ur}(\mu_{p^\infty}) \arrow[d] \arrow[dr] & \\
K^{ur} \arrow[dr] & K(\mu_{p^\infty}) \arrow[d, "\text{subgroup of } \mathbb{Z}_p^\times"] \\
 & K
\end{tikzcd}$$

Key : Convert $K(\mu_{p^\infty})$ to equal characteristic fields.