Introduction to p-adic Hodge theory 2

——— Grothendieck's monodromy theorem, Toolbox of Newton polygon.

<u>Recall</u>: K finite extension of $\mathbb{Q}_p$.

$\ell \neq p$, $\iota_\ell : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$

$$\begin{array}{ccccccccc} 1 & \to & I_K & \to & \mathrm{Gal}_K & \to & \mathrm{Fr}_K^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \| & & \cup\!| & & \cup\!| & & \\ 1 & \to & I_K & \to & W_K & \xrightarrow{\mathrm{val}} & F_K^{\mathbb{Z}} & \to & 1 \end{array}$$

$t_\ell : I_K \twoheadrightarrow I_K^{\mathrm{tame}} \simeq \hat{\mathbb{Z}}(1) \to \mathbb{Z}_\ell$

└ depending on a system of roots of unity $\zeta_{\ell^{r+1}}^{\ell} = \zeta_{\ell^r}$, denoted by $\underline{\zeta}$.

- Let $\rho : W_K \to GL_n(\bar{\mathbb{Q}}_\ell)$ be a continuous representation

<u>naïve idea</u>: $W_K \xrightarrow{\rho} GL_n(\bar{\mathbb{Q}}_\ell) \overset{\iota_\ell}{\simeq} GL_n(\mathbb{C})$ but this is not continuous.

Issue: ℓ-part of the tame inertia $\to GL_n(\bar{\mathbb{Q}}_\ell)$ $\rightsquigarrow$ N linear operator

<u>Lemma 1</u> (1) $\exists$ a finite extension E of $\mathbb{Q}_\ell$ (inside $\bar{\mathbb{Q}}_\ell$) s.t. $\mathrm{Im}(\rho) \subseteq GL_n(E)$.

(2) $\exists\, g \in GL_n(E)$ s.t. $g \cdot \rho(I_K) g^{-1} \subseteq GL_n(\mathcal{O}_E)$.

This works for all profinite groups in place of I_K.

Proof: (1) Uses Baire's category theory

(2) $\rho : I_K \to GL_n(E) \supseteq GL_n(\mathcal{O}_E)$. $\rightsquigarrow$ $\rho^{-1}(GL_n(\mathcal{O}_E)) =: H$ is open in I_K

So I_K/H is finite; write $I_K = \coprod_{j=1}^{m} h_j H$

Then $\Lambda' := \sum_{j=1}^{m} h_j(\mathcal{O}_E^{\oplus n})$ is a lattice stable under I_K-action.

Let $g :=$ change of basis matrix from $\mathcal{O}_E^{\oplus n}$ to Λ'

Then $g \cdot \rho(I_K) g^{-1} \subseteq GL_n(\mathcal{O}_E)$. $\square$

WLOG, $\rho : W_K \to GL_n(\mathcal{O}_E)$ $\quad V := E^{\oplus n}$

<u>Lemma 2</u>: $\rho(P_K)$ is finite. In particular, $\exists\, K''/K$ finite extension s.t. $\rho(P_{K''}) = \{1\}$.

Proof: Note that P_K is a pro-p group. $\Rightarrow \rho(P_K)$ is pro-p.

Consider $\rho(P_K) \subseteq GL_n(\mathcal{O}_E) \overset{\lambda}{\twoheadrightarrow} GL_n(k_E)$

$\ker\lambda = 1+\varpi_E M_n(\mathcal{O}_E)$ is a pro-ℓ group $\Rightarrow \rho(P_K)\cap\ker\lambda=\{1\}$

$\Rightarrow \rho(P_K)\longrightarrow GL_n(k_E)$ is injective $\Rightarrow \rho(P_K)$ is finite. $\square$

<u>Lemma 3</u> $\exists$ a finite extension K'/K s.t. $\rho(I_{K'})$ is unipotent, i.e. all (generalized) eigenvalues are 1.

Proof. Take K'' as in Lemma 2. $\Rightarrow \rho: W_{K''}/P_{K''}\longrightarrow GL_n(\mathcal{O}_E)$.

Let $Fr_{K''}$ = a geometric Frobenius element, $\tau_{K''}$ = generator of I_K^{tame}. $q'':=\#k''$

$\Rightarrow Fr_{K''}\tau_{K''}^{q}Fr_{K''}^{-1}=\tau_{K''}$.

$\Rightarrow \{\text{Eigenvalues of } \tau_{K''}\} = \{\text{Eigenvalues of } \tau_{K''}^{q}\}$

$\Rightarrow$ All eigenvalues of $\tau_{K''}$ are (q''^m-1)th root of unity for some $m\le n$.

After replacing K'' by K', can assume $\tau_{K'}=\tau_{K''}^{\prod(q''^m-1)}$. ✓ $\square$

<u>Proposition</u>. $\exists$ a unique <u>nilpotent</u> operator $N\in \mathrm{End}(V)$ and a finite ext'n K'/K

s.t. for $\tau\in I_{K'}$, $\rho(\tau)=\exp(t_\ell(\tau)N)$ $\qquad t_\ell: I_K\longrightarrow \mathbb{Z}_\ell(1)\overset{\text{choice of uniformizer}}{\simeq}\mathbb{Z}_\ell$

Moreover, for any geometric Frobenius element Fr_K, we have

$$\rho(Fr_K)^{-1}\cdot N\rho(Fr_K)=q\cdot N. \qquad (*)$$

Proof: Take K' as above, take $N:=\frac{1}{m}\log(\tau_{K'}^{m})$ for m suff. divisible.

Then $(*)$ follows from $Fr_K^{-1}\tau_{K'}^{q}Fr_K=\tau_{K'}$. $\square$

<u>Theorem</u> (Grothendieck's monodromy theorem)

Fix a geom. Frob. element Fr_K and t_ℓ (depending on a trivialization $\mathbb{Z}_\ell(1)\simeq\mathbb{Z}_\ell$)

Then there's an equivalence of categories

$$\left\{\begin{matrix}\rho: W_K\longrightarrow GL_n(E)\\ \text{cont. representations}\end{matrix}\right\}\longleftrightarrow\left\{(r,N)\;\middle|\;\begin{matrix}r: W_K\longrightarrow GL_n(E)\text{ cont. rep'n for discrete top on } E\\ N\in M_n(E)\text{ s.t. } r(g)Nr(g)^{-1}=q^{-\mathrm{val}(g)}N\end{matrix}\right\}$$

Given ρ, $r(Fr_K^a\cdot x):=\rho(Fr_K^a\cdot x)\cdot\exp(-t_\ell(x)\cdot N)$.

Proof: left as an exercise. $\square$

<u>Definition</u> A <u>Weil-Deligne representation</u> $V/\mathbb{C}$ consists of (r, N):

(1) $r: W_K \to GL_n(V)$ a continuous representation ($\Rightarrow r(I_K)$ is finite)

(2) $N \in \mathrm{End}(V)$ s.t. $r(g) N r(g)^{-1} = q^{-\mathrm{val}(g)}(N)$

- So composing the above equivalence with $\otimes_{E,\iota_\ell}\mathbb{C}$ gives the functor $WD(-)$

<u>Remarks</u>. (1) The isomorphism class of $WD(\rho)$ is independent of the choice of Fr_K and $\underline{\zeta}$.

(2) A fancy way to write N is: $N: V \to V(-1)$ a homomorphism

$$N g v = g N v \cdot q^{\mathrm{val}(g)}$$

One way to think of this: action induces $\underset{\substack{\parallel \\ V(1)}}{\mathrm{Lie}\, I_K \otimes V} \to V$

Back to independence-of-ℓ question over K:

<u>Conjecture</u> (Independence of ℓ) Let K be a finite extension of $\mathbb{Q}_p$ and let X be a proj. sm var$/K$.

Then (1) the action of any Frobenius Fr_K on $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple.

(2) $WD(H^i_{et}(X_{\overline{K}}, \overline{\mathbb{Q}}_\ell))$ is independent of ℓ

Would be interesting to have a p-adic version.

This semester: will construct a functor

$$WD: \{\text{de Rham representation } \rho: G_K \to GL_n(\overline{\mathbb{Q}}_p)\} \longrightarrow \{\text{Weil-Deligne representations}\}$$

<u>Toolbox of Newton polygon</u>

<u>Notation</u>: K = complete valued field $|\cdot|: K \to \mathbb{R}_{\geq 0}$, $v(\cdot) = \log_p |\cdot| : K^\times \to \mathbb{R}$

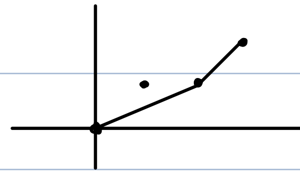
<u>Definition</u> For a formal Laurent series $f(x) = \sum_{n\in\mathbb{Z}} a_n t^n$, $a_n \in K$,

draw its Newton polygon $NP(f) :=$ lower convex hull of

$(n, v(a_n)) \quad n \in \mathbb{Z}$. (if it exists)

(If f has only finitely many terms, just draw a finite polygon.)

E.g. $f(x) = 1 + px + px^2 + p^2x^3$



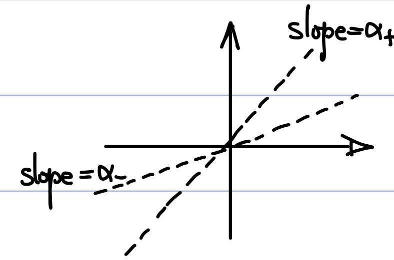
(In practice, we will assume some control over $v_p(a_n)$ so that $f(t)\cdot g(t)$ makes sense.)

<u>Theorem</u> (Vertex factorization). Suppose that $NP(f)$ has a vertex P_0, then we have a decomposition

$$f(t) = g(t)\cdot h(t), \text{ where } NP(g) = \text{part of } NP(f) \text{ to the left of } P_0$$
$$NP(h) = \text{part of } NP(f) \text{ to the right of } P_0.$$

More precisely, (if P_0 is at $x=m$, we may mult. f by $a_m^{-1}t^{-m}$)

we may assume $P_0 = (0,0)$ and $a_0 = 1$.



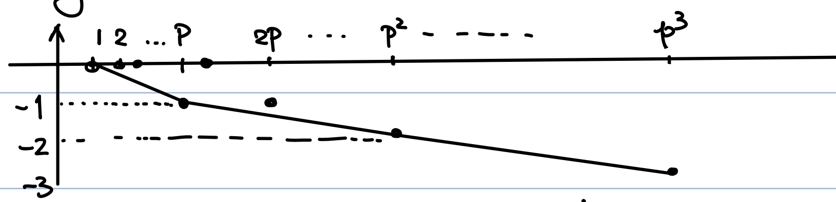
Suppose $\exists\, \alpha_+ > \alpha_-$ s.t. $NP(f)_{x\geq 0} \geqslant \alpha_+\cdot x$, $NP(f)_{x\leq 0} \geqslant \alpha_-\cdot x$.

Then there's a unique decomposition $f(t) = g(t)\cdot h(t)$

$$\text{for } g(t) \in (1+\mathfrak{m}_K) + t\cdot K[\![t]\!],\ h(t) \in 1 + t^{-1}K[\![t^{-1}]\!]$$

such that $NP(g) = NP(f)_{x\geq 0}$, $NP(h) = NP(f)_{x\leq 0}$

<u>Example</u>: $\log(1+t) = t - \frac{t^2}{2} + \cdots = \sum_{n\geq 1} \frac{(-1)^n}{n} t^n$.



<u>Note</u>: For each p^nth root of unity $\zeta_{p^n}^j$, $\log \zeta_{p^n}^j = \frac{1}{p^n}\cdot \log\left(\zeta_{p^n}^j\right)^{p^n} = \frac{1}{p^n}\log 1 = 0$

So $\zeta_{p^n}^j - 1$ is a zero of $\log(1+t)$

Expect $$Q_n(t) = \prod_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \left(t - (\zeta_{p^n}^j - 1)\right) = \prod_{j\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\left((t+1) - \zeta_{p^n}^j\right) = \frac{(1+t)^{p^n}-1}{(1+t)^{p^{n-1}}-1}$$

the p^nth cyclotomic polynomial $= t^{p^n-p^{n-1}} + \cdots + p$

<u>Fact</u>: $\log(1+t) = t \cdot \prod_{n\geq 1} \frac{Q_n(t)}{p}$ ← corresponding to the n^{th} segment of $NP(\log(1+t))$

Proof: Will only prove existence; uniqueness left as a homework.

Set $g^{(0)}(t) = 1 + \sum_{m\geq 1} a_m t^m$, $h^{(0)}(t) = 1 + \sum_{n\geq 1} a_{-n} t^{-n}$,

$$f - g^{(0)} h^{(0)} = \left(\sum_{n\geq 1} a_{-n} t^{-n} + 1 + \sum_{m\geq 1} a_m t^m\right) - \left(1 + \sum_{m\geq 1} a_m t^m\right)\left(1 + \sum_{n\geq 1} a_{-n} t^{-n}\right)$$

$$= -\left(\sum_{m\geq 1} a_m t^m\right)\cdot\left(\sum_{n\geq 1} a_{-n} t^{-n}\right)$$

<u>Lemma</u> Suppose $\tilde{g} = \sum_{m\geq 1} g_m t^m \in t\cdot K[\![t]\!]$, $\tilde{h} = \sum_{n\geq 1} h_{-n} t^{-n} \in t^{-1} K[\![t]\!]$

and suppose $NP(\tilde{g}) \geq NP(f)_{x>0}$ and $NP(\tilde{h}) \geq NP(f)_{x<0}$

Then $NP(\tilde{g}\cdot\tilde{h}) \geq NP(f) + (\alpha_+ - \alpha_-)$.

Proof: $\tilde{g}\tilde{h} = \sum_{\ell\in\mathbb{Z}} \left(\sum_{m-n=\ell} g_m h_{-n}\right) t^\ell$

WLOG $\ell \geq 0$ and $m-n=\ell$,

$$v(g_m h_{-n}) \geq NP(f)_{x=m} + NP(f)_{x=-n}$$

$$\geq NP(f)_{x=\ell} + \alpha_+(m-\ell) + \alpha_-\cdot(-n)$$

$$= NP(f)_{x=\ell} + n\cdot(\alpha_+ - \alpha_-) \overset{\text{since } n\geq 1}{\geq} NP(f)_{x=\ell} + (\alpha_+ - \alpha_-) \quad \square$$

Back to the proof of vertex factorization: Put $\delta := \alpha_+ - \alpha_- > 0$

we will construct $g^{(s)}(t) = \sum_{m\geq 0} g_m^{(s)} t^m$, $h^{(s)}(t) = 1 + \sum_{n\geq 1} h_{-n}^{(s)} t^{-n}$,

s.t. ① $NP(g^{(s+1)} - g^{(s)}) \geq NP(f)_{x\geq 0} + \frac{s+2}{2}\delta$

② $NP(h^{(s+1)} - h^{(s)}) \geq NP(f)_{x<0} + \frac{s+1}{2}\delta$

③ $NP(f - g^{(s)} h^{(s)})_{x\geq 0} \geq NP(f)_{x\geq 0} + \frac{s+2}{2}\delta$

$NP(f - g^{(s)} h^{(s)})_{x<0} \geq NP(f)_{x<0} + \frac{s+1}{2}\delta$

slight "twist" to make the proof work.

Then $g := \lim_{s\to\infty} g^{(s)}$, $h := \lim_{s\to\infty} h^{(s)}$ satisfies the theorem.

③ is satisfied for $s=0$ by the lemma above

Now if $g^{(s)}, h^{(s)}$ are given & satisfy ③,

Write $f - g^{(s)}h^{(s)} = \sum_{m\geq 0} b_m t^m + \sum_{n\geq 1} b_{-n} t^{-n}$, say const coeff of $g^{(s)}$ is $c_0 \in \mathcal{O}_K^\times$

Put $g^{(s+1)} = g^{(s)} + \sum_{m\geq 0} b_m t^m$ and $h^{(s+1)} = h^{(s)} + c_0^{-1} \sum_{n\geq 1} b_{-n} t^{-n}$, They satisfy ①②

Now check ③: $f - g^{(s+1)}h^{(s+1)}$

$$= g^{(s)}h^{(s)} + \sum_{m\geq 0} b_m t^m + \sum_{n\geq 1} b_{-n} t^{-n} - \Big(g^{(s)} + \sum_{m\geq 0} b_m t^m\Big)\Big(h^{(s)} + c_0^{-1}\sum_{n\geq 1} b_{-n} t^{-n}\Big)$$

$$= \Big(\sum_{m\geq 1} b_m t^m\Big)(h^{(s)} - 1) + b_0 \cdot (h^{(s)} - 1) + \Big(\sum_{n\geq 1} b_{-n} t^{-n}\Big)(c_0^{-1} g^{(s)} - 1) + \Big(\sum_{m\geq 1} b_m t^m\Big)\Big(\sum_{n\geq 1} b_{-n} t^{-n}\Big) + b_0\Big(\sum_{n\geq 1} b_{-n} t^{-n}\Big)$$

p-adic valuation versus

$NP(f)_{x\geq 0}$	$\frac{s+2}{2}\delta + \delta$	∞	$\frac{s+1}{2}\delta + \delta$	$\frac{s+1}{2}\delta + \frac{s+2}{2}\delta + \delta$	∞	$\geqslant \frac{s+3}{2}\delta$
$NP(f)_{x<0}$	$\frac{s+2}{2}\delta + \delta$	$\frac{s+2}{2}\delta$	$\frac{s+1}{2}\delta + \delta$	$\frac{s+1}{2}\delta + \frac{s+2}{2}\delta + \delta$	$\frac{s+2}{2}\delta$	$\geqslant \frac{s+2}{2}\delta$

This proves the vertex factorization. $\square$