Introduction to p-adic Hodge theory 5

—— Lubin-Tate construction of local class field theory

Recall $K/\mathbb{Q}_p$ finite extension, uniformizer ϖ_K, residue field $\mathbb{F}_q$

$$\mathcal{F}_\varpi := \left\{ \text{power series } f(x) \text{ in } \mathcal{O}_K[\![X]\!] \text{ s.t. } \begin{array}{l} f(x) = \varpi x + \deg \geq 2 \text{ terms} \\ f(x) \equiv x^q \mod \varpi \end{array} \right\}$$

Key technical lemma. Let $f(x), g(x) \in \mathcal{F}_\varpi$. Let A be a ϖ-adic complete $\mathcal{O}_K$-algebra.

Let $a_1, \cdots, a_n \in A$. Then there is a unique power series $\phi \in A[\![X_1, \cdots, X_n]\!]$

s.t. $\begin{cases} \phi(X_1, \cdots, X_n) = a_1 X_1 + \cdots + a_n X_n + \text{terms of } \deg \geq 2 \\ f(\phi(X_1, \cdots, X_n)) = \phi(g(X_1), \cdots, g(X_n)). \end{cases}$

Proof: Write $\phi_1(X_1, \cdots, X_n) = a_1 X_1 + \cdots + a_n X_n$.

We prove by induction on r that there's a unique polynomial $\phi_r(X_1, \cdots, X_n)$ of $\deg \leq r$

s.t. $\begin{cases} \phi_r(X_1, \cdots, X_n) = \phi_1 + \text{terms of } \deg \geq 2 \\ f(\phi_r(X_1, \cdots, X_n)) = \phi_r(g(X_1), \cdots, g(X_n)) + \text{terms of } \deg \geq r+1. \end{cases}$

* ϕ_1 is okay b/c $f(\phi_1(X_1, \cdots, X_n)) - \phi_1(g(X_1), \cdots, g(X_n)) \equiv \pi \phi_1(X_1, \cdots, X_n) - \phi_1(\pi X_1, \cdots, \pi X_n)$ mod $\deg \geq 2$.

Suppose we have ϕ_r, want to find $\phi_{r+1} = \phi_r + Q$ for Q homogeneous poly of deg $r+1$.

Want $f(\phi_r(X_1, \cdots, X_n) + Q) \equiv \phi_r(g(X_1), \cdots, g(X_n)) + \underbrace{Q(g(X_1), \cdots, g(X_n))}$ mod $\deg \geq r+2$

$$f(\phi_r(X_1, \cdots, X_n) + Q) = f(\phi_r(X_1, \cdots, X_n)) + \varpi Q + \text{sth. in deg} \geq r+2$$

$$Q(g(X_1), \cdots, g(X_n)) = \varpi^{r+1} \cdot Q(X_1, \cdots, X_n) + \text{sth. in deg} \geq r+2$$

We can solve this iff ϖ divides $\phi_r(g(X_1), \cdots, g(X_n)) - f(\phi_r(X_1, \cdots, X_n))$ ← of $\deg \geq r+1$

$$\equiv \phi_r(X_1^p, \cdots, X_n^p) - \phi_r(X_1, \cdots, X_n)^p \equiv 0 \mod \varpi. \quad \square$$

Corollary (1) For each $f \in \mathcal{F}_\varpi$, there exists a unique $\mathcal{O}_K$-formal module F_f for which

$$[\varpi]_{F_f}(X) = f(X).$$

(2) For each $f, g \in \mathcal{F}_\varpi$, there exists a unique isomophism $\phi(X) = X + \deg \geq 2$ terms

s.t. $F_g(\phi(X), \phi(Y)) = \phi(F_f(X,Y))$.

Proof: (1) By Key Lemma, $\exists!\ F_f(X,Y) \in X + Y + (\text{terms of } \deg \geq 2) \in \mathcal{O}_K[\![X,Y]\!]$

s.t. $f(F_f(X,Y)) = F_f(f(X), f(Y))$

(Setting $Y=0$, we have $f(F_f(X,0)) = F_f(f(X), 0)$ $(*)$

Such $F_f(X,0)$ is unique & $F_f(X,0) = X$ satisfies $(*)$.)

So $F_f(X,0) = X$

By uniqueness, $F_f(X,Y) = F_f(Y,X)$.

For every $a \in \mathcal{O}_K$, $\exists!\ [a]_f(X) = aX + \cdots$ s.t.

$$[a]_f(f(X)) = f([a]_f(X))$$

$*$ We have $F([a]_f(X), [a]_f(Y)) = [a]_f(F(X,Y))$ b/c both hands commute with $f(X)$

$[a]_f(X) = f(X)$ by uniqueness.

This defines a formal $\mathcal{O}_K$-module.

(2) Applying key lemma gives the isomouphism. $\phi(X) = X + (\text{terms of } \deg \geq 2)$.

Easy to check all needed commutativity. $\square$

- First, fix $f \in \mathcal{F}_\pi$

Let $\Lambda_{\varpi,n} :=$ zero of $[\varpi^n]_f(X) = \{ z \in K^{alg} ;\ [\varpi^n](z) = 0 \}$

Theorem For each $n > 0$, put $K_{\varpi,n} := K(\Lambda_{\varpi,n})$

(1) For each n, $K_{\varpi,n}/K$ is totally ramifield of degree $(q-1)q^n$

(2) The action of $(\mathcal{O}_K/\varpi^n)^\times$ on $\Lambda_{\varpi,n}$ induces an automorphism of $K_{\varpi,n}$, and hence an isom.

$$(\mathcal{O}_K/\varpi^n)^\times \xrightarrow{\sim} \mathrm{Gal}(K_{\varpi,n}/K)$$

(3) ϖ is a norm from $K_{\varpi,n}$.

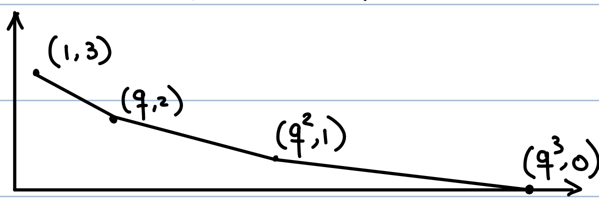
(4) $K_{\varpi,n}$ is independent of the

Proof: (1) The leading term of $f(X) = [\varpi]_f(X) \approx \varpi X + X^q$

$\Rightarrow [\varpi^2]_f(X) = f \circ f \approx \varpi(\varpi X + X^q) + (\varpi X + X^q)^q \approx \varpi^2 X + \varpi X^q + X^{q^2}$

We prove that $NP([\varpi^r]_f)$ has vertices:

$$(1, r), (q, r-1), (q^2, r-2), \dots, (q^r, 0)$$



Proof by induction: Suppose $NP([\varpi^r]_f)$ is as above

$$f([\varpi^r]_f) = \underbrace{\varpi \cdot [\varpi^r]_f}_{\text{move up by 1}} + \underbrace{[\varpi^r]_f^q}_{\text{scaling } q \text{ times}} + \underbrace{\varpi \cdot [\varpi^r]_f^2 \cdot \text{other} \cdots}_{\text{scaling twice + move up by 1.}} \checkmark$$

$Q_n(X) := [\varpi^n]_f(X) / [\varpi^{n-1}]_f(X) =$ polynomial whose NP is a straight line

$$= X^{q^{n-1}(q-1)} + \cdots + \varpi.$$

$\Rightarrow Q_n(X)$ is irreducible by Eisenstein criterion & $K_{\varpi,n}$ is totally ramified$/K$.

(2) For each n, fix a zero ϖ_n of $Q_n(X)$, so that $[\varpi]_f(\varpi_n) = \varpi_{n-1}$.

For $a \in \mathcal{O}_K^\times$, $[a]_f(X)$ sends a zero of $Q_n(X)$ to another zero of $Q_n(X)$

This gives a group homomorphism $\psi: (\mathcal{O}_K/\varpi^n)^\times \to \mathrm{Aut}(K_{\varpi,n}/K)$

Will show that $\ker(\psi) = \{1\}$.

First, if $a \in \mathcal{O}_K^\times$ and $\bar{a} \neq 1$, $[a]_f(\varpi_n) \equiv a\varpi_n \mod \varpi_n^2$. so $\psi(a) \neq \mathrm{id}$

If $1+\varpi^r \in \ker\psi \Rightarrow [1+\varpi^r]_f(\varpi_n) = [1]_f(\varpi_n) \Rightarrow [\varpi^r]_f(\varpi_n) = 0$.

This is only possible if $r \geq n$.

(3) ϖ is the norm of ϖ_n.

(4) If $g \in \mathcal{F}_\pi$, $\exists\, \phi$ isom of formal groups $F_f \xrightarrow{\sim} F_g$

then if ϖ_n is a zero of $\dfrac{[\varpi^n]_f(x)}{[\varpi^{n-1}]_f(x)} \Rightarrow \phi(\varpi_n)$ is a zero of $\dfrac{[\varpi^n]_g(x)}{[\varpi^{n-1}]_g(x)}$. $\square$

<u>Corollary</u>:

$$\begin{tikzcd}[column sep=small]
 & K^{ab} \arrow[dl, dash] \arrow[dr, dash] & \\
\bigcup K_{\varpi,n} = K_\varpi \arrow[dr, dash] & & K^{ur} \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Under the reciprocity law $\widehat{K^\times} \xrightarrow{\simeq} \mathrm{Gal}(K^{ab}/K)$

$\mathrm{Gal}(K^{ab}/K_\varpi)$ corresponds to $\varpi^{\hat{\mathbb{Z}}}$.

Next, we verify $K_\varpi \cdot K^{ur} = K_{\varpi'} \cdot K^{ur}$ for two uniformizers ϖ and ϖ'.

Fix $f \in \mathcal{F}_\varpi$ and $f' \in \mathcal{F}_{\varpi'}$. Put $\varpi' = u \cdot \varpi$ for $u \in \mathcal{O}_K^\times$

<u>Key point of the proof above</u>: ① $\underbrace{(\mathfrak{m}_{K^{ab}}, +_f) \circlearrowleft \mathcal{O}_K^\times}$; action defined over $\mathcal{O}_K$

This structure is independent of the choice of f, up to isom.

② The $\mathcal{O}_K^\times$-action is the <u>same</u> as the reciprocity $\mathrm{Gal}(K_\varpi/K) \simeq \mathcal{O}_K^\times$-action

<u>Goal</u>: Define an isom $(\mathfrak{m}_{\widehat{K^{ab}}}, +_{f'}) \xrightarrow[\sim]{\phi} (\mathfrak{m}_{\widehat{K^{ab}}}, +_{f'})$ over $\widehat{\mathcal{O}_{K^{ur}}}$ with compatible

$\phi \in \mathcal{O}_{K^{ur}}[[X]]$ $\qquad\qquad$ $\mathrm{Gal}(K^{ab}/K^{ur}) = \mathcal{O}_K^\times$-action

- Compatibility w/ mult. by $[\varpi']$: $\phi([u]_f(f(X))) = f'(\phi(X))$ $\quad (*)$

Rmk: For this equation, the proof of the key lemma does <u>NOT</u> work.

- <u>Need</u> $\phi(z_n) \in K_{\varpi'}$

$$\begin{tikzcd}[column sep=small]
 & K^{ab} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K_\varpi \arrow[dr, dash] & K_{\varpi'} \arrow[d, dash] & K^{ur} \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

$\mathrm{Art}(\varpi')^{-1}(\phi(z_n)) = \phi(z_n)$

$\Leftrightarrow \mathrm{Art}(\varpi)^{-1}\phi(z_n) = [u]_f\,\phi(z_n)$

$\Leftrightarrow \sigma(\phi(z_n)) = [u]_f\,\phi(z_n)$

σ is the arithmetic Frobenius

So we require $\sigma(\phi(x)) = [u]_f(\phi(x))$ $\quad (**)$

- Combining $(*)(**) \Rightarrow \sigma(\phi(f(x))) = f'(\phi(x))$

Condition on the linear term of $\phi(x) = \lambda X + \cdots$

$$\sigma(\lambda)\cdot\varpi = \varpi'\cdot\lambda$$

<u>Theorem</u> (1) There exists $\lambda \in \widehat{\mathcal{O}}_{K^{ur}}^{\times}$ s.t. $\sigma(\lambda)/\lambda = u$ (such λ is unique up to $\mathbb{Z}_p^{\times}$)

Proof: This follows from Hilbert 90' theorem:

$$H^1(\mathrm{Gal}(K^{ur}/K), \widehat{\mathcal{O}}_{K^{ur}}^{\times}) = 0 \Leftarrow \begin{cases} H^1(\mathrm{Gal}(K^{ur}/K), \overline{\mathbb{F}}_q^{\times}) = 0 \\ H^1(\mathrm{Gal}(K^{ur}/K), \underbrace{\varpi^m\mathcal{O}_{K^{ur}}/\varpi^{m+1}\mathcal{O}_{K^{ur}}}_{=\overline{\mathbb{F}}_q}) = 0 \end{cases}$$

(2) Using the $\lambda \in \widehat{\mathcal{O}}_{K^{ur}}^{\times}$, there's a unique $\phi(X) = \lambda X + (\text{terms of deg} \geq 2)$

s.t. $\phi(f([u]_f(X))) = f'(\phi(x))$ commuting ϖ-action & provides uniqueness.

For this ϕ, we must have

- $\phi([u]_f(X)) = \sigma(\phi(X))$ descent data
- $\phi(F_f(X,Y)) = F_{f'}(\phi(X),\phi(Y))$
- $\forall a \in \mathcal{O}_K$, $\phi([a]_f(X)) = [a]_{f'}(\phi(X))$

(3) Given a zero ϖ_n of $[\varpi^n]_f(X)/[\varpi^{n-1}]_f(X)$, $\phi(\varpi_n)$ is a zero of $[\varpi'^n]_{f'}(x)/[\varpi'^{n-1}]_{f'}(x)$.

Moreover, for the arithmetic Frobenius σ from K^{ur},

$$\sigma(\phi(\varpi_n)) = \phi([u]_f(\varpi_n)) = [u]_{f'}(\phi(\varpi_n))$$

Note: It is expected that $\phi(\varpi_n)$ is fixed by $\varpi' = \varpi\cdot u = \sigma^{-1}\circ[u]_{f'}$ ✓

- <u>Useful structures on Lubin-Tate formal groups</u>

- K finite extension of $\mathbb{Q}_p$ with uniformizer ϖ and residue field $\mathbb{F}_q$

Fix $f \in \mathcal{F}_\varpi$ and we have a formal $\mathcal{O}_K$-module F_f.

Write $\varpi_0 = \varpi$, for $m \geq 1$, fix a uniformizer ϖ_m of $[\varpi^m]_f(X) / [\varpi^{m-1}]_f(X)$

so that $[\varpi]_f(\varpi_m) = \varpi_{m-1}$

$$v(\varpi_m) = \frac{1}{q^{m-1}(q-1)}$$

<u>Note</u>: $[\varpi]_f(\varpi_m) \equiv \varpi_m \cdot \varpi + \varpi_m^q \pmod{\varpi_m^2 \cdot \varpi}$

In particular, $\varpi_{m-1} \equiv \varpi_m^q \bmod (\varpi)$; except when $m=1$, $\varpi_1^{q-1} \equiv \varpi_0 \pmod{\varpi}$

- Over K, we in fact have an isom $\phi : F_f \xrightarrow{\sim} \hat{\mathbb{G}}_a$

formal $\mathcal{O}_K$-module where $F(X,Y) = X+Y$
$[a]_F(X) = a \cdot X$

For this, we expect (on points)

$$\phi(z) = \frac{1}{\varpi^n} \phi\left([\varpi^n]_f(z)\right)$$

<u>Theorem</u>
$$\mathrm{Log}_f(X) := \lim_{n \to +\infty} \frac{1}{\varpi^n} \left([\varpi^n]_f(X)\right) = X \cdot \prod_{n \geq 1} \left(\frac{[\varpi^n]_f(X)}{\varpi \cdot [\varpi^{n-1}]_f(X)}\right)$$

$$= X \prod_{z \in \bigcup_m \Lambda_{f,m}} \left(1 - \frac{X}{z}\right)$$

defines an isom $F_f \xrightarrow{\sim} \hat{\mathbb{G}}_a$

<u>Note</u>: $\dfrac{[\varpi^n]_f(X)}{\varpi \cdot [\varpi^{n-1}]_f(X)} = 1 + \cdots + \dfrac{X^{q^{n-1}(q-1)}}{\varpi}$ has NP with slopes $-\dfrac{1}{q^{n-1}(q-1)}$

<u>Example</u>: When $K = \mathbb{Q}_p$, $f = (1+X)^p - 1$, $\mathrm{Log}_f(X) = \log(1+X)$.