Introduction to p-adic Hodge theory 8

—— Tilting equivalence

Good reference: Kedlaya, New methods for (φ,Γ)-modules

Recall: K complete valued field of mixed characteristic $(0,p)$; $v_K(\varpi)\in(0,v_K(p))$

Assume K is perfectoid i.e. $\Phi: \mathcal{O}_K/\varpi \twoheadrightarrow \mathcal{O}_K/\varpi \quad x\longmapsto x^p$

$\rightsquigarrow \mathcal{O}_{K^\flat} := \varprojlim_{\Phi} \mathcal{O}_K/\varpi \overset{\eta}{\dashrightarrow} \varprojlim_{\Phi} \mathcal{O}_K$, $\varpi^\flat := (\varpi, \varpi^{1/p}, \dots)$

$K^\flat = \mathcal{O}_{K^\flat}\left[\frac{1}{\varpi^\flat}\right]$

Theorem There is a natural surjective ring homomorphism

$$\theta: W(\mathcal{O}_{K^\flat}) \twoheadrightarrow \mathcal{O}_K$$

induced by the map $\mathcal{O}_{K^\flat} \to \mathcal{O}_K/p \quad x\longmapsto \bar{x}_0$

Explicitly, $\theta\left(\sum_{n\geq 0} p^n [x^{(n)}]\right) = \sum_{n\geq 0} p^n \cdot \eta(x^{(n)})_0$

Note: θ is surjective b/c it is so modulo p.

Weak topology on $W(\mathcal{O}_{K^\flat})$: A sequence $y_1, y_2, \dots$ with $y_i = (y_i^{(0)}, y_i^{(1)}, \dots)$ $y_i^{(j)}\in \mathcal{O}_{K^\flat}$

converges if $\forall m\geq 1, \varepsilon>0, \exists N$ s.t. if $n>N$,

$\forall j\in\{0,\dots,m\}, \quad \left|y_n^{(j)} - y_{n+1}^{(j)}\right|_{K^\flat} < \varepsilon$

Remark: $W(\mathcal{O}_{K^\flat})$ admits the following universal property:

If $D \overset{\theta_D}{\twoheadrightarrow} \mathcal{O}_K$ is any surjective continuous homomorphism s.t. D is complete for $\ker\theta_D$-topology, & p-adic topology

then there exists a unique continuous homomorphism

$$\begin{tikzcd} W(\mathcal{O}_{K^\flat}) \arrow[rr, two heads, "\theta"] \arrow[dr, dashed, "\alpha"'] & & \mathcal{O}_K \\ & D \arrow[ur, "\theta_D"'] & \end{tikzcd}$$

Proof: For this, by universal property of Witt vectors, it is enough to construct

$$\begin{tikzcd} \mathcal{O}_{K^\flat} \arrow[rr, "\bar\theta"] \arrow[dr, dashed, "\bar\alpha"'] & & \mathcal{O}_K/p \\ & D/pD \arrow[ur, "\bar\theta_D"'] & \end{tikzcd}$$

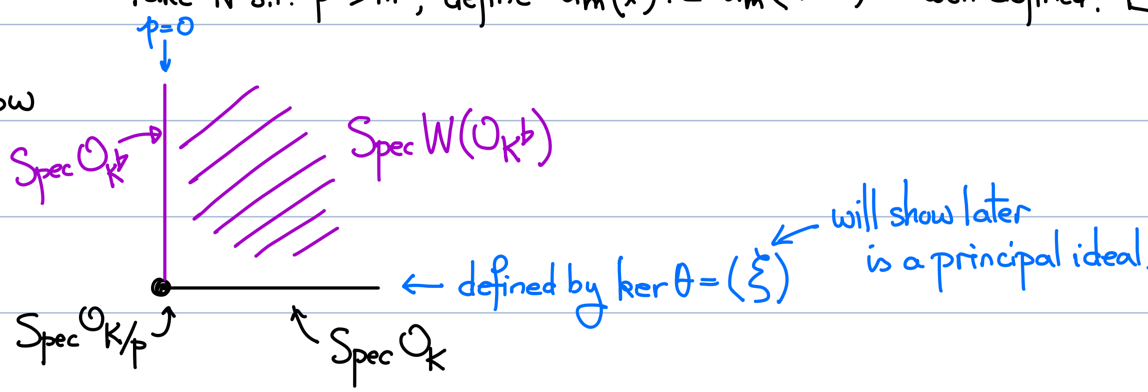
Since D is complete for $\ker\theta_D$-adic topology, $\overset{\bar D}{\overbrace{D/pD}}$ is complete for $\ker\bar\theta_D$-adic topology

Suffices to construct

$$\begin{tikzcd} \mathcal{O}_{K^\flat} \arrow[rr, "\bar\theta"] \arrow[dr, dashed, "\bar\alpha_m"'] & & \mathcal{O}_K/p \\ & \bar D/(\ker\bar\theta_D)^m \arrow[ur] & \end{tikzcd}$$

Take N s.t. $p^N > m$, define $\bar\alpha_m(x) := \bar\alpha_m(\widetilde{x^{1/p^N}})^{p^N}$ well-defined. $\square$

Somehow



<u>Study $\ker\theta$ / Recover $\mathcal{O}_K$ from $\mathcal{O}_{K^\flat}$</u> (normalize valuation s.t. $v_p(p)=1$)

Write $\mathbb{E} := K^\flat$, only view $\mathbb{E}$ as a complete discrete valued field of char p that is perfect. (& that $v(\mathbb{E}^\times) \supseteq \mathbb{Z}$.) $\Rightarrow v_{\mathbb{E}}(\mathbb{E}^\times) \supseteq \mathbb{Z}[\frac{1}{p}]$

<u>Definition</u> An element $\xi \in W(\mathcal{O}_{\mathbb{E}})$ is called <u>primitive</u> if

$$\xi = [\varpi_0] + p\cdot u \text{ for } \begin{cases} \varpi_0 \in \mathcal{O}_{\mathbb{E}}, \text{ s.t. } v_{\mathbb{E}}(\varpi_0) = 1 \\ u \in W(\mathcal{O}_{\mathbb{E}})^\times \end{cases}$$

(Exercise: if $v \in W(\mathcal{O}_{\mathbb{E}})^\times$, ξ primitive $\Rightarrow v\cdot\xi$ is primitive.)

<u>Theorem</u>: If $z \in W(\mathcal{O}_{\mathbb{E}})$ is a primitive element, then

① $\underset{\mathcal{O}_{\mathbb{E}}^\#}{\underset{\shortparallel}{W(\mathcal{O}_{\mathbb{E}})/(z)}}$ is the valuation ring of a perfectoid completed valuation field of mixed char.

② $\mathcal{O}_{\mathbb{E}}^{\#,\flat} \cong \mathcal{O}_{\mathbb{E}}$

③ Conversely, if K is a perfectoid complete valued field of mixed char,

$\theta : W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$ is generated by a primitive element.

In short, we have a natural equivalence of categories

$$\left\{ \begin{array}{c} \text{mixed char perfectoid} \\ \text{complete valued fields} \end{array} \right\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{char } p \text{ perfectoid complete valued field } \mathbb{E} \\ \text{equipped with a primitive principal ideal } (\xi) \end{array} \right\}$$

$$K \longmapsto K^\flat, \ker\left(W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K\right)$$

$$W(\mathcal{O}_{\mathbb{E}})/(\xi) \longleftarrow (\mathbb{E}, (\xi))$$

(Toy model: $W(\mathcal{O}_{K^\flat}) \rightsquigarrow \mathcal{O}_K[\![T]\!]$ T plays the role of p

$\forall$ uniformizer $\varpi \in \mathcal{O}_K$ $\quad \mathcal{O}_K[\![T]\!]/(\varpi) \simeq \mathcal{O}_K$

The toy model can talk about "Gaussian norm" we do the same for $W(\mathcal{O}_{\mathbb{E}})$.

Technical definition For each $r > 0$, define the Gauss valuation $v_r(\cdot)$ on $W(\mathcal{O}_{\mathbb{E}})$ to be

$$v_r\left(\sum_{n \geq 0} p^n [x_n]\right) := \inf_{n \geq 0} \left(n + v_{\mathbb{E}}(x_n) \cdot r\right)$$

This is the analogue of $\frac{1}{r}$-Gauss valuation on $\mathcal{O}_K[\![T]\!]$.

$$v_\infty\left(\sum_{n \geq 0} p^n [x_n]\right) := \inf_{n \geq 0} \left(v_{\mathbb{E}}(x_n)\right)$$

Claim: v_r is a valuation for all $r \in (0, +\infty]$, (namely the strong triangle inequality holds).

Proof: $v_r([x] \cdot [y]) = v_r([x]) + v_r([y])$.

Need to show $v_r([x] + [y]) \geqslant \min\left(v_r(x), v_r(y)\right)$ $\quad (*)$

This is because $[x] + [y] = [S_0(x,y)] + p \cdot [S_1(x,y)] + p^2 \cdot [S_2(x,y)] + \cdots$,

where $S_i(x,y) \in \mathbb{F}_p[x^{1/p^i}, y^{1/p^i}]^{\deg = 1}$

So $(*)$ ✓. $\qquad \square$

Technical definition: An element $x = \sum_{n\geq 0} p^n [\bar{x}_n] \in W(\mathcal{O}_{\mathbb{E}})$ is <u>stable</u> if $\forall n$, $v_{\mathbb{E}}(a_n) \geq v_{\mathbb{E}}(a_0)$.

Lemma. An element $x \in W(\mathcal{O}_{\mathbb{E}})$ is stable

$\iff v_{\mathbb{E}}(\bar{x}_0) = v_\infty(x)$.

$\iff x =$ prod of a Teichmüller & a unit $\in W(\mathcal{O}_{\mathbb{E}})$.

Proof: $[x] \cdot \sum_{n\geq 0} p^n [y_n] = \sum_{n\geq 0} p^n [x y_n]$

note $\sum_{n\geq 0} p^n [y_n]$ is a unit $\iff v_{\mathbb{E}}(y_0) = 0$.

Lemma. For any primitive $\xi \in W(\mathcal{O}_{\mathbb{E}})$, every class in $W(\mathcal{O}_{\mathbb{E}})/(\xi)$ is represented by a stable element of $W(\mathcal{O}_{\mathbb{E}})$

Proof: Write $\xi = [\varpi_0] + pu$ for $\varpi_0 \in \mathcal{O}_{\mathbb{E}}$, $u \in W(\mathcal{O}_{\mathbb{E}})^\times$

Given $x \in W(\mathcal{O}_{\mathbb{E}})$, $x = [\bar{x}_0] + p[\bar{x}_1] + p^2[\bar{x}_2] + \cdots$

(Imagine $x = [x_0] + p[x_1]$ with $v(x_1) \ll v(x_0)$

we should write $x = [x_0] + p[x_1] - \xi \cdot (u^{-1}[x_1])$

$= [x_0] - [\varpi_0] \cdot [x_1] \cdot u^{-1}$! This is okay...)

We define inductively, $x_0 = x$, for $x_\ell = [\bar{x}_{\ell,0}] + p[\bar{x}_{\ell,1}] + p^2[\bar{x}_{\ell,2}] + \cdots$

Put $y_\ell = [\bar{x}_{\ell,1}] + p[\bar{x}_{\ell,2}] + \cdots$

Define $x_{\ell+1} = [\bar{x}_{\ell,0}] - [\varpi_0] \cdot u^{-1} \cdot y_\ell$

$$= x_\ell - p y_\ell - [\varpi_0] \cdot u^{-1} \cdot y_\ell = x_\ell - y_\ell u^{-1} \underbrace{(pu + [\varpi_0])}_{= \xi}$$

At each step: • either $v_{\mathbb{E}}(\bar{x}_{\ell,0}) \leq v_{\mathbb{E}}(\bar{x}_{\ell,n})$ for all $n > 0$ $\Rightarrow x_\ell$ is stable

• or $v_{\mathbb{E}}(\bar{x}_{\ell,0}) > v_{\mathbb{E}}(\bar{x}_{\ell,n})$ for some $n > 0$

then $v_\infty(y_\ell) = v_\infty(x_\ell)$

<u>Note</u>: $v_\infty\left([\varpi_0] \cdot u^{-1} \cdot y_\ell\right) = 1 + v_\infty(y_\ell)$.

So if $v_\infty(\bar{x}_{\ell,0}) < v_\infty(x_\ell) - 1$, then $x_{\ell+1}$ is stable

or if $v_\infty(\bar{x}_{\ell,0}) \geq v_\infty(x_\ell) - 1 \Rightarrow v_\infty(x_{\ell+1}) \leq v_\infty(x_\ell) - 1$.

↑ such step stops at finite times

Lemma. If $x \in W(\mathcal{O}_{\mathbb{E}})$ is stable & $x = \xi \cdot y$ for some primitive ξ and $y \in W(\mathcal{O}_{\mathbb{E}})$.

then $x = 0$.

Proof: By looking at first term, $v_{\mathbb{E}}(\bar{x}_0) = 1 + v_{\mathbb{E}}(\bar{y}_0) \geq 1 + v_{\infty}(y)$

But $\underset{\underset{v_\infty(\bar{x}_0)}{\|}}{v_\infty(x)} = v_\infty(\xi) + v_\infty(y) = v_\infty(y)$. Contradiction

Lemma. If $\xi \in W(\mathcal{O}_{\mathbb{E}})$ is primitive and $x, y \in W(\mathcal{O}_{\mathbb{E}})$ are stable s.t. $x \equiv y \mod (\xi)$

Then $v_{\mathbb{E}}(\bar{x}_0) = v_{\mathbb{E}}(\bar{y}_0)$.

Proof: Suppose $v_{\mathbb{E}}(\bar{x}_0) < v_{\mathbb{E}}(\bar{y}_0)$. Put $w = x - y$,

Then $v_{\mathbb{E}}(\bar{w}_0) = \min\{v_{\mathbb{E}}(\bar{x}_0), v_{\mathbb{E}}(\bar{y}_0)\} = v_{\mathbb{E}}(\bar{x}_0) = \min\{v_\infty(x), v_\infty(y)\} \leq v_\infty(w)$

So w is stable but divisible by ξ. $\Rightarrow w = 0$ by previous lemma.

Proof of Theorem ① ② We define a valuation $v_R(-)$ on $R := W(\mathcal{O}_{\mathbb{E}})/(\xi)$ as follows

$\forall\ y \in W(\mathcal{O}_{\mathbb{E}})/(\xi)$, pick a stable lift $\tilde{y} \in W(\mathcal{O}_{\mathbb{E}})$ and define

$$v_R(y) = v_\infty(\tilde{y}) = v_{\mathbb{E}}(\bar{\tilde{y}}_0).$$

This is independent of the lifts.

- $v_R(-)$ is multiplicative, b/c product of stable elements is stable.

  R is complete w.r.t. $v_R(-)$

  $v_R(x+y) \geq \min(v_R(x), v_R(y))$

  b/c if $w = \tilde{x} + \tilde{y}$ stable lift, $v_R(w) \geq v_\infty(w) \geq \min(v_\infty(\tilde{x}), v_\infty(\tilde{y}))$

  $= \min(v_R(x), v_R(y))$.

- There's a natural isom $\mathcal{O}_{\mathbb{E}}/(\bar{\varpi}_0) = W(\mathcal{O}_{\mathbb{E}})/(p, \xi) = R/pR$

- R is the valuation ring of $R[\frac{1}{p}]$, a perfectoid mixed char field

  (need: if $x, y \in W(\mathcal{O}_{\mathbb{E}})$ are stable and $v_\infty(x) \leq v_\infty(y)$, then y is div. by x

  This is because we can write $x = [\bar{x}_0] \cdot u$ and $y = [\bar{y}_0] \cdot v$ for $u, v \in W(\mathcal{O}_{\mathbb{E}})^\times$

  $\Rightarrow x = y \cdot [\frac{\bar{x}_0}{\bar{y}_0}] \cdot uv^{-1}$.)

- $K := R[\frac{1}{p}]$ is perfectoid b/c Φ is surj on $\mathcal{O}_{\mathbb{E}}/(\bar{\varpi}_0) \Rightarrow \Phi$ surj on R/pR.
- $K^\flat \simeq \mathbb{E}$ b/c $\varprojlim_{\Phi} R/pR \simeq \varprojlim_{\Phi} \mathcal{O}_{\mathbb{E}}/(\bar{\varpi}_0) \simeq \mathcal{O}_{\mathbb{E}}$.
- $\mathrm{Ker}(\theta : W(\mathcal{O}_{\mathbb{E}}) \to \mathcal{O}_K)$ is generated by ξ.

<u>Proof of Thm ③</u>. $\theta : W(\mathcal{O}_{K^\flat}) \twoheadrightarrow \mathcal{O}_K$

Recall that $v_{K^\flat}(\mathcal{O}_{K^\flat}) = v_K(\mathcal{O}_K)$, so $\exists\, \bar{\varpi}_0 \in \mathcal{O}_{K^\flat}$ s.t. $v_{K^\flat}(\bar{\varpi}_0) = v(p) = 1$.

Then $\theta([\bar{\varpi}_0]) = p \cdot v$ for some $v \in \mathcal{O}_K^\times$,

take any $u \in W(\mathcal{O}_{K^\flat})$ s.t. $\theta(u) = v$, $\Rightarrow u \in W(\mathcal{O}_{K^\flat})^\times$.

Note: $\xi = [\bar{\varpi}_0] + pu$ is primitive $\in \ker\theta$.

<u>Claim</u> $\ker(\theta) = (\xi)$.

This is because $(W(\mathcal{O}_{K^\flat})/(\xi))[\frac{1}{p}] \twoheadrightarrow \mathcal{O}_K[\frac{1}{p}] = K$ are perfectoids

so must be isomorphism.