Introduction to p-adic Hodge theory 9

— Tilting equivalence for finite extensions.

Last time There is an equivalence of categories

$$\left\{\begin{array}{c}\text{mixed char perfectoid}\\ \text{complete valued fields } K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{char } p \text{ perfect complete valued field } \tilde{\mathbb{E}}\\ \text{with a primitive principal ideal } (\xi)\end{array}\right\}$$

with valuation $v_K(-)$ with valuation $v_{\tilde{\mathbb{E}}}(-)$.

$$K \longmapsto K^\flat,\ \ker\left(W(\mathcal{O}_{K^\flat}) \xrightarrow{\theta} \mathcal{O}_K\right)$$

$$W(\mathcal{O}_{\tilde{\mathbb{E}}})/(\xi)[\tfrac{1}{p}] \longleftarrow (\tilde{\mathbb{E}}, (\xi)) \qquad \left(\text{write } \theta([\varpi_0]) =: \varpi_0^\#\right)$$

- An intriguing remark: If F is a complete valued field ext'n of $\tilde{\mathbb{E}}$, we can use same ξ to get a field extension $F^\#$ of K.

Example. $K = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$

$$\mathcal{O}_{K^\flat} = \varprojlim_{\Phi} \mathcal{O}_K/(\zeta_p - 1) \simeq \varprojlim_{\Phi} \mathcal{O}_K \ni \varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \leftarrow \text{fix such elt.}$$

Consider $\theta : W(\mathcal{O}_{K^\flat}) \twoheadrightarrow \mathcal{O}_K$ but $[\varepsilon] - 1 = [\varepsilon - 1] + p \cdot \cdots\cdots$

$[\varepsilon] - 1 \longmapsto 0$ $\quad \varepsilon - 1 = (0, \zeta_p - 1, \zeta_{p^2} - 1, \cdots)$ yet $v_{\tilde{\mathbb{E}}}(\varepsilon - 1) = \frac{p}{p-1}$

A primitive element is typical taken to be $\xi = \dfrac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{p-1/p}] \in \ker\theta$

note: $1 + \varepsilon^{1/p} + \cdots + \varepsilon^{\frac{p-1}{p}} = (1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1}, \underbrace{\zeta_p + \zeta_{p^2} + \zeta_{p^2}^2 + \cdots + \zeta_{p^2}^{p-1}}_{(\zeta_{p^2}^{p+1} - \zeta_{p^2})/(\zeta_{p^2} - 1)}, \cdots)$

So $\xi = [\varpi] + pu$ for some $u \in W(\mathcal{O}_{K^\flat})$

$\theta(\xi) = 0 \Rightarrow \theta(u) \in \mathcal{O}_K^\times \Rightarrow u \in W(\mathcal{O}_{K^\flat})^\times$.

its valuation is $\frac{1}{p-1} - \frac{1}{p(p-1)} = \frac{1}{p}$

Now, let K be a perfectoid complete valued field of mixed char.

<u>Theorem</u> There is an equivalence of categories

$$\{\text{finite étale algebras over } K\} \longleftrightarrow \{\text{finite étale algebras over } K^\flat\}$$

$$L \longmapsto L^\flat = \mathcal{O}_{L^\flat}\left[\tfrac{1}{\varpi}\right] = \left(\varprojlim_{\Phi}\left(\frac{\mathcal{O}_{L^\flat}}{\varpi \mathcal{O}_{L^\flat}}\right)\right)\left[\tfrac{1}{\varpi}\right]$$

$$\mathcal{O}_F^\sharp := W(\mathcal{O}_F) \Big/ \underset{\substack{\| \\ (\xi)}}{\ker\theta} \cdot W(\mathcal{O}_F) \longleftarrow\!\!\shortmid F$$

In particular, $\mathrm{Gal}(K^{sep}/K) \simeq \mathrm{Gal}(K^{\flat, sep}/K^\flat)$.

<u>Difficulty</u>: L/K finite $\Rightarrow$ $L^\flat/K^\flat$ finite is difficult.

Need to go from $F/K^\flat$ to $F^\sharp/K$.

Proof: ① If F is a finite extension of $K^\flat$, then $F^\sharp := \left(W(\mathcal{O}_F)/(\xi)\cdot W(\mathcal{O}_F)\right)[\frac{1}{p}]$ is a finite ext'n of K.

Then $[F : K^\flat] = [F^\sharp : K]$

② Under the untilt, if F is an algebraically closed, $F^\sharp$ is also algebraically closed.

For ①,

$$\begin{tikzcd} & & \widetilde{F}^\sharp \arrow[dll, dash] \\ F^\sharp \arrow[d, dash] & & \\ K \arrow[uurr, dash] & & \end{tikzcd} \longleftrightarrow \begin{tikzcd} & & \widetilde{F} \arrow[dll, dash] \\ F \arrow[d, dash] & & \\ K^\flat \arrow[uurr, dash, "G"'] & & \end{tikzcd}$$

$\widetilde{F}$ = Galois closure of $K^\flat$

By passing to Galois closure, suffices to prove $[\widetilde{F} : K^\flat] = [\widetilde{F}^\sharp : K]$

Consider the averaging $\sum_{g\in G}(-)$ functor

$$\widetilde{F}^\sharp = \frac{W(\mathcal{O}_{\widetilde{F}})[\frac{1}{p}]}{\xi\cdot W(\mathcal{O}_{\widetilde{F}})[\frac{1}{p}]} \xrightarrow{\sum_{g\in G}(-)} \frac{W(\mathcal{O}_{\widetilde{F}})^G[\frac{1}{p}]}{\xi\cdot W(\mathcal{O}_{\widetilde{F}})[\frac{1}{p}] \cap W(\mathcal{O}_{\widetilde{F}})^G[\frac{1}{p}]} = \frac{W(\mathcal{O}_{\widetilde{F}^G})[\frac{1}{p}]}{\xi W(\mathcal{O}_{\widetilde{F}^G})[\frac{1}{p}]}$$

$$\widetilde{F}^{\sharp, G} \dashrightarrow \widetilde{F}^{G,\sharp}$$

characteristic zero, so factors through.

But $\widetilde{F}^{G,\sharp}$ is invariant under G-action $\Rightarrow$ $\widetilde{F}^{G,\sharp} \subseteq \widetilde{F}^{\sharp, G}$. $\Rightarrow$ $\widetilde{F}^{G,\sharp} = \widetilde{F}^{\sharp, G}$

Moreover, G acts faithfully on $\tilde{F}^{\#}$.

This is b/c if $g \in G$ moves $x \in \mathcal{O}_{\tilde{F}}$, i.e. $v_{\tilde{F}}(gx - x) = M$

$$\exists\ n \text{ s.t. } v_{\tilde{F}}(gx^{1/p^n} - x^{1/p^n}) = \frac{M}{p^n} \cdot v_{\tilde{F}}(gx - x) < 1$$

$$\theta(g([x^{1/p^n}])) - \theta([x^{1/p^n}]) = \theta([gx^{1/p^n}] - [x^{1/p^n}]) = \theta(\underbrace{[gx^{1/p^n} - x^{1/p^n}]}_{\text{valuation} < 1} + p \cdot \boxed{\ }) \neq 0$$

So g acts nontrivially on $\mathcal{O}_{\tilde{F}^{\#}}$.

By Galois theory, $[\tilde{F}^{\#} : \tilde{F}^{\#, G}] = |G| = [\tilde{F} : \tilde{F}^{G}]$. ✓

For ②, let $P(T) = T^d + \cdots \in \mathcal{O}_{F^{\#}}[T]$ be an arbitrary polynomial.

We want to show F algebraically closed $\Rightarrow P(T)$ has a zero in $\mathcal{O}_{F^{\#}}$.

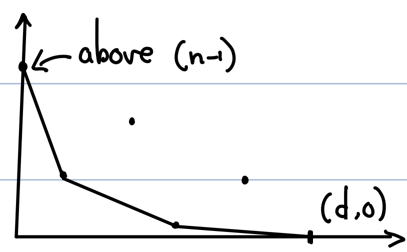
Will construct a series $x = x_0 + x_1 + x_2 + \cdots$ with $x_i \in \mathcal{O}_{F^{\#}}$, s.t.

$$\left.\begin{array}{l} (1)\ v_{F^{\#}}(x_n) \geq \dfrac{n-1}{d} \\ (2)\ v_{F^{\#}}(P(x_0 + \cdots + x_n)) \geq n \end{array}\right\} \Rightarrow x \text{ converges to a zero of } P(T)$$

Take $x_0 = 0$. Suppose (2) holds for x_{n-1} and we want to find x_n

Write $Q(T) := P(x_0 + \cdots + x_{n-1} + T) = \sum Q_i T^i$. If $Q_0 = 0$, nothing need to do

NP(Q)



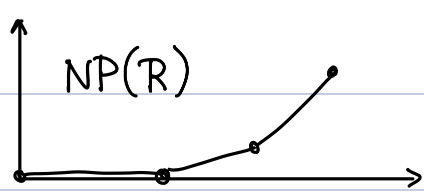
otherwise, for the NP(Q),

$-$ leftest slope $=: s \geq \dfrac{n-1}{d}$

↑ say of width d'

Since $v_{\#}(F^{\#, \times}) = v(F^{\times})$, $\exists\ u \in \mathcal{O}_F$ s.t. $v(u) = s$

Consider instead $R(T) := \frac{1}{u^d} \cdot P(x_0 + \cdots + x_{n-1} + u \cdot T) = \sum R_i T^i$



Need to find a zero of R, or rather just need a zero of R mod $\varpi^{\#}$.

Note $\mathcal{O}_{F^{\#}} / \varpi_0^{\#} = W(\mathcal{O}_F) / (\xi, [\varpi_0]) = W(\mathcal{O}_F) / (p, [\varpi_0]) = \mathcal{O}_F / \varpi_0$

$\nwarrow R(T)$ $\nwarrow R^\flat(T)$

$\exists\ \tilde{R}^\flat(T) \in \mathcal{O}_F[T]$ lifting $R^\flat(T)$

$\tilde{R}^\flat(T)$ has a zero $y_n \in \mathcal{O}_F^\times$

$\rightsquigarrow [y_n]$ is a zero of $R(T)$ mod $\varpi^\#$

$\rightsquigarrow u\cdot[y_n]$ is a zero of $Q(T)$ mod $\mathrm{val}^? n \geq d\, v(u)+1 \geq n$

& $v_\#(u\cdot[y_n]) \geq \frac{n-1}{d}$ ✓

Final words: Note that $K^{\flat,alg} = \bigcup_{F/K^\flat \text{ finite}} F$ is dense in $\widehat{K^{\flat,alg}}$

$\Rightarrow \underbrace{\bigcup_{F/K^\flat \text{ fin}} F^\#}_{}$ is dense in $\widehat{K^{\flat,alg}}^{\#}$.

$L :=$ $\uparrow$ union of finite extensions

$\left.\begin{array}{l}\Rightarrow L \subseteq K^{alg} \text{ and} \\ \text{But } \widehat{K^{\flat,alg}}^{\#} \text{ is alg. closed} \Rightarrow K^{alg} \subseteq \widehat{K^{\flat,alg}}^{\#} = \hat{L}\end{array}\right\} \Rightarrow L = K^{alg}$ $\square$

Corollary: If K is a perfectoid complete valued field,

$\mathrm{Gal}(K^{alg}/K) \simeq \mathrm{Gal}(K^{\flat,alg}/K^\flat)$ is canonically isomorphic!

Remark: This is stronger than saying $\mathrm{Gal}_K \simeq \mathrm{Gal}_{K^\flat}$ because

the algebraic closure of K is not unique, so Gal_K only makes sense up to conjugation

(This is okay for Galois representations (up to isomorphism))

Tilting equivalence is "completely canonical".

- More discussion on $W(\mathcal{O}_{K^\flat})$ with $K = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ "notation from Fontaine era"

$\mathcal{O}_{K^\flat} = \mathbb{F}_p[[\varepsilon-1]]^{perf,\wedge} = \tilde{\mathbb{E}}^+_{\mathbb{Q}_p} \supseteq \mathbb{E}^+_{\mathbb{Q}_p} = \mathbb{F}_p[[\varepsilon-1]]$

$K^\flat = \mathcal{O}_{K^\flat}[\frac{1}{\varepsilon-1}] = \tilde{\mathbb{E}}_{\mathbb{Q}_p} \supseteq \mathbb{E}_{\mathbb{Q}_p} = \mathbb{F}_p((\varepsilon-1))$

$\mathbb{C}_p^\flat \stackrel{\text{later}}{=} (K^{\flat,alg})^\wedge = \tilde{\mathbb{E}} \supseteq \mathbb{E} = \mathbb{F}_p((\varepsilon-1))^{sep}$

$$W(\mathcal{O}_{K^\flat}) = W\left(\mathbb{F}_p[[\varepsilon-1]]^{perf,\wedge}\right) = \widetilde{\mathbb{A}}^+_{\mathbb{Q}_p} \supseteq \mathbb{A}^+_{\mathbb{Q}_p} = \mathbb{Z}_p[[\underbrace{[\varepsilon]-1}_{=T}]] = \mathbb{Z}_p[[T]]$$

$$\mathbb{A}_{\mathbb{Q}_p} = \mathbb{Z}_p((T))^{\wedge,p\text{-adic}}, \quad \mathbb{B}_{\mathbb{Q}_p} = \mathbb{A}_{\mathbb{Q}_p}[\tfrac{1}{p}]$$

Fontaine era's philosophy: $W(\mathcal{O}_{K^\flat})$ is too "non-noetherian", yet $\mathbb{A}^+_{\mathbb{Q}_p}$ is quite noetherian.

$$\Gamma := \mathrm{Gal}\left(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p\right) = \mathbb{Z}_p^\times \curvearrowright \widetilde{\mathbb{E}}_{\mathbb{Q}_p}, \mathbb{E}_{\mathbb{Q}_p}, \widetilde{\mathbb{A}}^{(+)}_{\mathbb{Q}_p}, \mathbb{A}^{(+)}_{\mathbb{Q}_p}$$

$$(\zeta_{p^n} \mapsto \zeta_{p^n}^\gamma) \longleftarrow\!\shortmid\ \gamma$$

__Note__: $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ $\quad \gamma(\varepsilon) = (1, \zeta_p^\gamma, \zeta_{p^2}^\gamma, \dots)$

$$\gamma([\varepsilon]) = [\varepsilon^\gamma] = [\varepsilon]^\gamma \qquad \varphi([\varepsilon]) = [\varepsilon^p] = [\varepsilon]^p \text{ Frobenius}$$

Typical variable in (φ,Γ)-module theory $T := [\varepsilon]-1$

then $\gamma(T) = (1+T)^\gamma - 1$, $\varphi(T) = (1+T)^p - 1$.

Basic setup for (φ,Γ)-module setup

$$\begin{tikzcd}
\widehat{\overline{\mathbb{Q}}_p} = \mathbb{C}_p \arrow[d, dash] \arrow[r, squiggly, "\text{tilt}"] & \widehat{K^{\flat,alg}} = \widetilde{\mathbb{E}} \arrow[d, dash] & \supset & \mathbb{E} = \mathbb{F}_p((\overline{T}))^{sep} \arrow[d, dash] & & \mathbb{A} = \widehat{\mathbb{A}^{ur}_{\mathbb{Q}_p}} \arrow[d, dash] \\
\widehat{\mathbb{Q}_p(\mu_{p^\infty})} = K \arrow[d, dash, "\Gamma"'] & K^\flat = \widetilde{\mathbb{E}}_{\mathbb{Q}_p} \arrow[loop below, "\Gamma"] & \supset & \mathbb{E}_{\mathbb{Q}_p} = \mathbb{F}_p((\overline{T})) \arrow[rr, squiggly] & & \mathbb{A}_{\mathbb{Q}_p} = \mathbb{Z}_p((T))^{\wedge,p\text{-adic}} \\
\mathbb{Q}_p & & & & &
\end{tikzcd}$$

$$\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty})\right) \simeq \mathrm{Gal}\left(K^{\flat,alg}/K^\flat\right) \simeq \mathrm{Gal}\left(\mathbb{F}_p((\overline{T}))^{sep}/\mathbb{F}_p((\overline{T}))\right) \simeq \mathrm{Gal}\left(\mathbb{A}/\mathbb{A}_{\mathbb{Q}_p}\right)$$

The same picture works for $\mathbb{Q}_p$ replaced by $W(k)[\frac{1}{p}]$ for k perfect