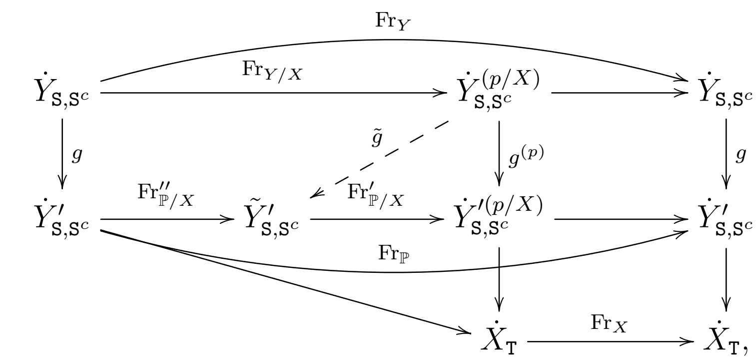
Erratum

It was communicated to us by Fred Diamond, Payman Kassaei, and Shu Sasaki that our proof of Proposition 3.19 in §4.9 is not correct as written. We thank them for pointing out this mistake. A major issue is the use of local model argument in the proof of statement (1) therein. In particular, it is not possible to trivialize $\mathcal{H}_\tau$ and $\mathcal{H}'_\tau$ as stated so that the trivialization $\mathcal{H}_\tau \simeq \mathcal{O}_{\mathcal{V},y}^{\wedge,\oplus 2}$ descends to $\mathcal{O}_{\mathcal{U},x}^{\wedge}$ (this is a statement we implicitly used).[1] We here give a proof of the statement (1) in §4.9 using a crystallization argument to avoid the local model language. A similar argument is given in [2, §7.1-7.2] with more details (in the case when p is assumed to be unramified for their purposes).

Keep the notation as in § 4 of the original paper. The idea of the new proof lies in (partially) untwisting the Frobenius factor morphism $g : \dot{Y}^{\circ}_{\mathtt{S},\mathtt{S}^c} \to \dot{Y}'_{\mathtt{S},\mathtt{S}^c}$ from Proposition 4.5, where we recall that $Y'_{\mathtt{S},\mathtt{S}^c} := \prod_{\tau\in\mathtt{S}\backslash\theta(\mathtt{S})} \mathbb{P}(\mathcal{H}_\tau)$ is a $\mathbb{P}^1$-power-bundle over $\dot{X}_{\mathtt{T}}$. To rigorously state the process of untwisting the partial Frobenius, we consider the following commutative diagram of relative Frobenius morphisms:

(1)



where Fr_X, $\mathrm{Fr}_{\mathbb{P}}$, and Fr_Y are the absolute Frobenius morphisms, the two squares on the right are Cartesian (which define $\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}$ and $\dot{Y}'^{(p/X)}_{\mathtt{S},\mathtt{S}^c}$), and $\mathrm{Fr}_{Y/X}$ and $\mathrm{Fr}'_{\mathbb{P}/X} \circ \mathrm{Fr}''_{\mathbb{P}/X}$ are two relative Frobenius morphisms. We define $\tilde{Y}'_{\mathtt{S},\mathtt{S}^c}$ so that both $\mathrm{Fr}'_{\mathbb{P}/X}$ and $\mathrm{Fr}''_{\mathbb{P}/X}$ are Frobenius factors and $\mathrm{Fr}'_{\mathbb{P}/X}$ is the Frobenius on those $\mathbb{P}^1$ fibers which are labeled by $\tau \in \mathtt{S}\backslash\theta(\mathtt{S})$ such that $\tau = \tau^{(\ell)}_{\mathfrak{p},j}$ with $\ell = 1$.

The following factorization result is the key.

Proposition 1. *Keep the notation as in the paper. There is a natural morphism $\tilde{g}$ as in* (1) *to make the diagram commute. Moreover, $\tilde{g}$ induces an* injection *on relative differentials.*

(2)
$$\tilde{g}^*\Omega^1_{\tilde{Y}'_{\mathtt{S},\mathtt{S}^c}/\dot{X}_{\mathtt{T}}} \to \Omega^1_{\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}/\dot{X}_{\mathtt{T}}}.$$

Let us first show that this proposition implies the needed statement (1) in § 4.9 of the original paper. Indeed, let $\eta'_{\mathtt{T}}$ denote the generic point of $\dot{X}_{\mathtt{T}}$ and $\eta_{\mathtt{T}}$ the geometric generic

[1] Indeed, suppose this is possible (say in the case when p is completely inert in $F/\mathbb{Q}$). Then we may choose an isomorphism between $\mathcal{O}^{\wedge}_{\mathcal{V},y} \cong \mathcal{O}[\![(u_\tau)_{\tau\in\mathtt{S}}, (v_\tau)_{\tau\in\mathtt{S}^c}]\!]$ such that, when $\tau \in \mathtt{S}$ (resp. $\tau \in \mathtt{S}^c$), the differential sheaf ω_τ (resp. ω'_τ) is generated by $u_\tau e^{(1)}_\tau + e^{(2)}_\tau$ (resp. by $e'^{(1)}_\tau + v_\tau e'^{(2)}_\tau$) in the given local basis of the trivialization. But this would imply that, modulo p, the differential sheaf ω_τ when $\tau \in \mathtt{S}^c$ is generated by $e^{(1)}_\tau + \tau(\varpi)v_\tau e^{(2)}_\tau \equiv e^{(1)}_\tau$, which would then imply the image of $Y_{\mathtt{S},\mathtt{S}^c}$ in $X_{\mathtt{T}}$ is contained in a subvariety of dimension $\#\mathtt{S}$ (namely ω_τ does not move), which contradicts for example Proposition 4.5.

point of $\dot{X}_{\mathtt{T}}$.[2] Then (2) implies that $\tilde{g}_{\eta'_{\mathtt{T}}} : \dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}} \to \tilde{Y}'_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}}$ is a Frobenius factor and

$$\tilde{g}^*\Omega^1_{\tilde{Y}'_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}}/\eta'_{\mathtt{T}}} \to \Omega^1_{\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}}/\eta'_{\mathtt{T}}} \tag{3}$$

is isomorphism. Let $\kappa_{\eta'_{\mathtt{T}}}$ denote the residue field at $\eta'_{\mathtt{T}}$, and then the function field of $\tilde{Y}'_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}}$ is injective to $\kappa_{\eta'_{\mathtt{T}}}(x_1,\dots,x_n)$. The injectivity of (3) implies that the function field of $\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c,\eta'_{\mathtt{T}}}$ is contained in $\kappa^{\mathrm{perf}}_{\eta'_{\mathtt{T}}}(x_1,\dots,x_n)$, where $\kappa^{\mathrm{perf}}_{\eta'_{\mathtt{T}}}$ is the perfection of $\kappa_{\eta'_{\mathtt{T}}}$, or equivalently the residue field at $\eta'_{\mathtt{T}}$. From this it is easy to see that the natural map

$$\tilde{g}^{\mathrm{red}}_{\eta_{\mathtt{T}}} : (\dot{Y}_{\mathtt{S},\mathtt{S}^c})^{\mathrm{red}}_{\eta_{\mathtt{T}}} \cong (\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c})^{\mathrm{red}}_{\eta_{\mathtt{T}}} \to (\tilde{Y}'_{\mathtt{S},\mathtt{S}^c})_{\eta_{\mathtt{T}}}$$

is an isomorphism, and therefore the map $g^{\mathrm{red}}_{\eta_{\mathtt{T}}} : (\dot{Y}_{\mathtt{S},\mathtt{S}^c})^{\mathrm{red}}_{\eta_{\mathtt{T}}} \to (\dot{Y}'_{\mathtt{S},\mathtt{S}^c})_{\eta_{\mathtt{T}}}$ is the base change of $\mathrm{Fr}'_{\mathbb{P}/X}$, namely the p-Frobenius on the factor labeled by $\tau = \tau_a$ with $a \equiv 1 \bmod e$, and isomorphism on all other factors.

Proof of Proposition 1. We first investigate the situation at $\tau \in \mathtt{S}\backslash\theta(\mathtt{S})$ (where the projective bundle is taken), we have the following commutative diagram of locally free coherent sheaves over any S-point of $\dot{Y}_{\mathtt{S},\mathtt{S}^c}$ (for a noetherian $\mathbb{F}_p$-scheme S):

$$\begin{array}{ccccc} \mathscr{H}_\tau & \xrightarrow{\ \psi^*_\tau\ } & \mathscr{H}'_\tau & \xrightarrow{\ \phi^*_\tau\ } & \mathscr{H}_\tau \\ \big\downarrow{\scriptstyle \mathrm{Ha}_\tau} & & \big\downarrow{\scriptstyle \mathrm{Ha}'_\tau} & & \big\downarrow{\scriptstyle \mathrm{Ha}_\tau} \\ \mathscr{H}^{(p)}_{\theta^{-1}\tau} & \xrightarrow{\ \psi^{(p)*}_{\theta^{-1}\tau}\ } & \mathscr{H}'^{(p)}_{\theta^{-1}\tau} & \xrightarrow{\ \phi^{(p)*}_{\theta^{-1}\tau}\ } & \mathscr{H}^{(p)}_{\theta^{-1}\tau}, \end{array} \tag{4}$$

where following the convention of this section set forth in the proof of Proposition 4.5, if $\tau = \tau_a$ with $a \not\equiv 1 \bmod e$, we remove all the Frobenius twists on the modules at $\theta^{-1}\tau$. The condition $\theta^{-1}\tau \in \mathtt{S}^c$ implies that $\psi^{(p)*}_{\theta^{-1}\tau}(\mathrm{Im}(\mathrm{Ha}_\tau)) = 0$, so $\mathrm{Im}(\mathrm{Ha}_\tau) = \mathrm{Im}(\phi^{(p)*}_{\theta^{-1}\tau})$.

Now, consider the situation of an S-point of $\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}$, then the composition $S \to \dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c} \to \dot{Y}_{\mathtt{S},\mathtt{S}^c} \to \dot{X}_{\mathtt{T}}$ factors through the Frobenius $\mathrm{Fr}_X : \dot{X}_{\mathtt{T}} \to \dot{X}_{\mathtt{T}}$. Then on this S, A admits a Frobenius antecedent $A^{(p^{-1})}$. In particular, $\mathscr{H}_\tau$ for each τ admits a canonical Frobenius untwist $\mathscr{H}^{(p^{-1})}_\tau$. Then the lower right part of the diagram (4) admits a Frobenius antecedent:

$$\begin{array}{ccccc} & & & & \mathscr{H}^{(p^{-1})}_\tau \\ & & & \overset{\zeta_\tau}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{Ha}_\tau} \\ \mathscr{H}_{\theta^{-1}\tau} & \xrightarrow{\ \psi^*_{\theta^{-1}\tau}\ } & \mathscr{H}'_{\theta^{-1}\tau} & \xrightarrow{\ \phi^*_{\theta^{-1}\tau}\ } & \mathscr{H}_{\theta^{-1}\tau}, \end{array} \tag{5}$$

in which $\mathrm{Im}(\mathrm{Ha}_\tau) = \mathrm{Im}(\phi^*_{\theta^{-1}\tau})$.

Claim: There is a natural functorial isomorphism

$$\zeta_\tau : \mathscr{H}'_{\theta^{-1}\tau} \xrightarrow{\cong} \begin{cases} \mathscr{H}^{(p^{-1})}_\tau & \text{when } \tau = \tau_a \text{ with } a \equiv 1 \bmod e \\ (\mathscr{H}^{(p^{-1})}_\tau)^{(p)} & \text{when } \tau = \tau_a \text{ with } a \not\equiv 1 \bmod e, \end{cases}$$

so that the diagram (5) commutes.

Our argument is inspired by [1, Lemma 2.3] (which is similar to the crystallization process in [2, Lemma 7.2.1]). We assume that $\tau = \tau^{(1)}_{\mathfrak{p},j} = \tau_a$ with $a \equiv 1 \bmod e$ (that is Ha_τ is

[2] This choice of notation is to be consistent with the original paper.

induced by a Verschiebung morphism as opposed to multiplication by ϖ), and the other case is similar. Consider the jth factor of contravariant Dieudonné module $\mathcal{H}^1_{\mathrm{cris}}(-)_{\mathfrak{p},j}$ of the $\mathfrak{p}$-divisible groups of abelian varieties over the crystalline site of S (relative to $\mathcal{O}$), we upgrade the diagram (5) above to a commutative diagram of crystals:

$$\begin{array}{ccccc} & & & & \widetilde{\mathscr{H}_\tau}^{(p^{-1})} \\ & & & \overset{\tilde{\zeta}_\tau}{\nearrow} & \big\downarrow \widetilde{\mathrm{Ha}}_\tau \\ \mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j} & \xrightarrow{\psi^*_{\mathfrak{p},j}} & \mathcal{H}^1_{\mathrm{cris}}(A')_{\mathfrak{p},j} & \xrightarrow{\phi^*_{\mathfrak{p},j}} & \mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j}, \end{array}$$

where $\widetilde{\mathscr{H}_\tau}^{(p^{-1})} := \varpi_{\mathfrak{p}}^{e-1}\mathcal{H}^1_{\mathrm{cris}}(A^{(p^{-1})})_{\mathfrak{p},j}$ and the morphism $\widetilde{\mathrm{Ha}}_\tau$ is induced by $V/\varpi_{\mathfrak{p}}^{e-1}$. This diagram is related to (5) in terms of natural isomorphisms $\mathcal{H}^1_{\mathrm{cris}}(A')_{\mathfrak{p},j}/p\mathcal{H}^1_{\mathrm{cris}}(A')_{\mathfrak{p},j} \cong H^1_{\mathrm{dR}}(A')_{\mathfrak{p},j}$ and $\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j}/p\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j} \cong H^1_{\mathrm{dR}}(A)_{\mathfrak{p},j}$. Through the morphisms $\widetilde{\mathrm{Ha}}_\tau$ and $\varphi^*_{\mathfrak{p},j}$, we may identify $\widetilde{\mathrm{Ha}}_\tau(\widetilde{\mathscr{H}_\tau}^{(p^{-1})})+\varpi_{\mathfrak{p}}\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j}$ and $\phi^*_{\mathfrak{p},j}\mathcal{H}^1_{\mathrm{cris}}(A')_{\mathfrak{p},j}+\varpi_{\mathfrak{p}}\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j}$ as subsheaves of $\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j}$ corresponding to the preimage of $\mathrm{Im}(\mathrm{Ha}_\tau) = \mathrm{Im}(\phi^*_{\theta^{-1}\tau})$ under the reduction

$$\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j} \twoheadrightarrow H^1_{\mathrm{dR}}(A)_{\mathfrak{p},j} \twoheadrightarrow \mathscr{H}_{\theta^{-1}\tau}.$$

This in particular gives rise to an isomorphism of $(\mathcal{O}_S)_{\mathrm{cris}}$-modules

$$\tilde{\zeta}_\tau : \phi_{\mathfrak{p},j}(\mathcal{H}^1_{\mathrm{cris}}(A')_{\mathfrak{p},j}) + \varpi_{\mathfrak{p}}\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j} \cong \widetilde{\mathrm{Ha}}_\tau(\widetilde{\mathscr{H}_\tau}^{(p^{-1})}) + \varpi_{\mathfrak{p}}\mathcal{H}^1_{\mathrm{cris}}(A)_{\mathfrak{p},j},$$

which induces an isomorphism $\zeta_\tau : \mathscr{H}'_{\theta^{-1}\tau} \xrightarrow{\cong} \mathscr{H}_\tau^{(p^{-1})}$, proving the Claim.

Now we define the morphism $\tilde{g} : \dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c} \to \tilde{Y}'_{\mathtt{S},\mathtt{S}^c}$ to send an S-point of $\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}$ discussed above to a point of $\tilde{Y}'_{\mathtt{S},\mathtt{S}^c}$ (over $\dot{X}_{\mathtt{T}}$ through an additional Frobenius morphism) represented by the line bundles:

- $\zeta_\tau(\omega'_{\theta^{-1}\tau}) \subseteq \mathscr{H}_\tau^{(p^{-1})}$ for $\tau \in \mathtt{S}\backslash\theta(\mathtt{S})$ such that $\tau = \tau^{(1)}_{\mathfrak{p},j}$,
- $\zeta_\tau(\omega'_{\theta^{-1}\tau}) = \phi^*_\tau(\mathscr{H}'_\tau) \subseteq \mathscr{H}_\tau$ for $\tau \in \mathtt{S}\backslash\theta(\mathtt{S})$ such that $\tau = \tau^{(i)}_{\mathfrak{p},j}$ with $i \neq 1$.

Such defined morphism clearly makes the diagram (1) commute. To see that the induced morphism (2) on differential forms is injective, we recall from [3, Theorem 2.9] and the proof of Proposition 3.3 that

- $\Omega^1_{\dot{X}_{\mathtt{T}}/\mathcal{O}}$ is a successive extension of $\omega_\tau \otimes (\mathcal{H}_\tau^{(p^{-1})}/\omega_\tau)$, corresponding to deformations of ω_τ inside $\mathscr{H}_\tau$.
- $\Omega^1_{\tilde{Y}'_{\mathtt{S},\mathtt{S}^c}/\dot{X}_{\mathtt{T}}}$ is the direct sum of $\mathcal{L}_\tau \otimes (\mathcal{H}_\tau/\mathcal{L}_\tau)$ for $\tau = \tau^{(\ell)}_{\mathfrak{p},j} \in \mathtt{S}\backslash\theta(\mathtt{S})$ with $\ell \neq 1$ and $\mathcal{L}_\tau \otimes (\mathcal{H}_\tau^{(p^{-1})}/\mathcal{L}_\tau)$ for $\tau = \tau^{(1)}_{\mathfrak{p},j} \in \mathtt{S}\backslash\theta(\mathtt{S})$, corresponding to deformations of the canonical line bundle $\mathcal{L}_\tau$ inside $\mathscr{H}_\tau$ and $\mathcal{L}_\tau$ inside $\mathscr{H}_\tau^{(p^{-1})}$, respectively.
- $\Omega^1_{\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}/\mathcal{O}}$ is a successive extension of $\omega_\tau \otimes (\mathcal{H}_\tau/\omega_\tau)$ for $\tau \notin \mathtt{S}^c$ and $\omega'_\tau \otimes (\mathcal{H}'_\tau/\omega'_\tau)$ for $\tau \in \mathtt{S}^c$, corresponding to deformations of ω_τ inside $\mathscr{H}_\tau$ and ω'_τ inside $\mathscr{H}'_\tau$, respectively.

The natural map $\tilde{g}^*\Omega^1_{\tilde{Y}'_{\mathtt{S},\mathtt{S}^c}/\dot{X}_{\mathtt{T}}} \to \Omega^1_{\dot{Y}^{(p/X)}_{\mathtt{S},\mathtt{S}^c}/\dot{X}_{\mathtt{T}}}$ is induced by sending

- the deformation of $\omega'_{\theta^{-1}\tau}$ inside $\mathscr{H}'_{\theta^{-1}\tau}$ to the deformation of $\mathcal{L}_\tau$ inside $\mathscr{H}_\tau$ for $\tau = \tau^{(\ell)}_{\mathfrak{p},j} \in \mathtt{S}\backslash\theta(\mathtt{S})$ with $\ell \neq 1$, and
- the deformation of $\omega'_{\theta^{-1}\tau}$ inside $\mathscr{H}'_{\theta^{-1}\tau}$ to the deformation of $\mathcal{L}_\tau$ inside $\mathscr{H}_\tau^{(p^{-1})}$ for $\tau = \tau^{(1)}_{\mathfrak{p},j} \in \mathtt{S}\backslash\theta(\mathtt{S})$.

By our construction above, this is an injection.[3] □

Additional typos in Proposition 4.5 The beginning part of the proof of Proposition 4.5 contains a few typos.

- Line 2 of the second paragraph, "For $\tau \in \mathtt{T} = \mathtt{S}\backslash\theta(\mathtt{S})$, namely $\tau \in \mathtt{S}$ and $\theta^{-1}\tau \in \mathtt{S}^c$" should be replaced by "For $\tau \in \mathtt{T} = \theta(\mathtt{S})\backslash\mathtt{S}$, namely $\tau \in \mathtt{S}^c$ and $\theta^{-1}\tau \in \mathtt{S}$".
- Line 2 after the diagram (4.5.2), $\theta^{-1}\tau \in \mathtt{S}^c$ should be $\theta^{-1}\tau \in \mathtt{S}$; and Line 5 $\tau \in \mathtt{S}$ should be $\tau \in \mathtt{S}^c$.

References

[1] J. de Jong, The moduli spaces of principally polarized abelian varieties with $\Gamma_0(p)$-level structure, *Journal of Algebraic Geometry* **2** (1993), 667–688.

[2] F. Diamond, P. Kassaei, and S. Sasaki, A mod p Jacquet-Langlands relation and Serre filtration via the geometry of Hilbert modular varieties: Splicing and dicing, `arXiv:2001.00530`.

[3] D. Reduzzi and L. Xiao, Partial Hasse invariants on splitting models of Hilbert modular varieties, *Ann. Sci. de l'ENS* **50** (2017), 579–607.

[3]Rigorously speaking, we need to work, Zariski locally, to trivialize the successive extensions consisting of $\Omega^1_{\dot{X}_{\mathtt{T}}/\mathcal{O}}$ and $\Omega^1_{\dot{Y}_{\mathtt{S},\mathtt{S}^c}/\mathcal{O}}$ above.